

Linear Algebra Notes — Lecture 9/10

The Transpose, the Inverse of a Matrix, and Elementary Matrices

Lecture pages 60–74 · Lay §2.1–2.2

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Overview

Last time (9/8 notes) we learned how to multiply matrices and which algebra rules products obey. This lecture adds two new operations on matrices, as announced in the “Plan for today” (p. 60):

1. **The transpose** A^T : turn rows into columns. The key rule is $(AB)^T = B^T A^T$.
2. **The inverse** A^{-1} : the matrix that “undoes” A . We get a formula for 2×2 matrices, rules such as $(AB)^{-1} = B^{-1}A^{-1}$, and a practical algorithm: row reduce $[A \ I_n]$ to $[I_n \ A^{-1}]$.

To prove that the algorithm works, the professor introduces *elementary matrices*: matrices that perform one row operation by left multiplication. With them he proves that an $n \times n$ matrix with n pivots is invertible.

Every blank and question in the notes is answered in a blue box. Typos are listed at the end.

Notation used throughout.

- a_{ij} : the entry of A in row i , column j (9/8 notes, p. 54). We also write $A = (a_{ij})$.
- I_n (or just I): the $n \times n$ identity matrix; e_i : its i -th column, the i -th standard basis vector (9/3 notes, p. 40).
- “Pivot”, “RREF”: as in row reduction (8/27–9/1 notes). RREF A is the reduced row echelon form of A .
- R_i : row i . “ $R_2 + 2R_1$ ” means “add 2 times row 1 to row 2”; the professor writes this as “ $+2 \textcircled{1}$ ” next to row 2.

1 The transpose (pp. 60–62)

The transpose swaps rows and columns; it respects sums and scalar multiples, and it reverses the order of a product: $(AB)^T = B^T A^T$.

Definition (transpose, p. 60)

If A is an $m \times n$ matrix, then A^T is the matrix whose i -th column is the i -th row of A . In terms of entries: the (j, i) entry of A^T is the (i, j) entry of A .

What is the size of A^T ? It is $n \times m$

A has m rows, and each row becomes a column of A^T , so A^T has m columns. Each row of A has n entries, so each column of A^T has n entries, i.e. A^T has n rows. Hence A^T is $n \times m$. Check with the example below: A is 2×3 and A^T is 3×2 . ✓

Example (p. 60). The rows $(1, 2, 3)$ and $(4, 5, 6)$ become the columns:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

Theorem (p. 60)

For matrices A, B of the same size and a scalar r :

1. $(A^T)^T = A$;
2. $(A + B)^T = A^T + B^T$;
3. $(rA)^T = rA^T$.

The notes give no proof; each rule is one line once you compare entries (an addition, following Lay §2.1, Theorem 3).

1. Transposing twice moves the (i, j) entry to position (j, i) and then back to (i, j) .

- The (j, i) entry of $(A + B)^T$ is the (i, j) entry of $A + B$, which is $a_{ij} + b_{ij}$. This is the (j, i) entry of A^T plus the (j, i) entry of B^T .
- The professor's margin note " $ra_{ij} \rightarrow ra_{ji}$ " is exactly this proof. The (i, j) entry of rA is ra_{ij} , and transposing moves it to position (j, i) . The (j, i) entry of rA^T is r times the (j, i) entry of A^T , which is also ra_{ij} .

Definition (symmetric matrix, p. 60)

A is **symmetric** if $A^T = A$.

A symmetric matrix must be square (by the size answer above, $A^T = A$ forces $n \times m = m \times n$, so $m = n$), and its entries are mirrored across the main diagonal: $a_{ij} = a_{ji}$. For example (an addition),

$$S = \begin{bmatrix} 1 & 7 & -2 \\ 7 & 0 & 4 \\ -2 & 4 & 5 \end{bmatrix} \text{ is symmetric: row } i \text{ equals column } i \text{ for } i = 1, 2, 3.$$

How many different entries can a symmetric $n \times n$ matrix contain? $\frac{n^2+n}{2} = \frac{n(n+1)}{2}$

The notes answer with a picture: split the matrix along the diagonal into a "top" triangle and a "bottom" triangle, and count

$$\frac{n^2 - n}{2} + n = \frac{n^2 + n}{2}.$$

Why: the n^2 entries consist of n diagonal entries and $n^2 - n$ off-diagonal ones. The off-diagonal entries come in mirror pairs $a_{ij} = a_{ji}$, half above the diagonal (top) and half below (bottom). The bottom is a copy of the top, so only the $\frac{n^2-n}{2}$ top entries and the n diagonal entries can be chosen freely.

Check with $n = 3$: $\frac{9+3}{2} = 6$, and S above is determined by the six numbers 1, 0, 5 (diagonal) and 7, -2, 4 (top). ✓

Theorem (p. 61)

If A is $m \times n$ and B is $n \times k$, then $(AB)^T = B^T A^T$.

Sizes first: B^T is $k \times n$ and A^T is $n \times m$, so $B^T A^T$ is defined and is $k \times m$, the same size as $(AB)^T$. (The product $A^T B^T$ would not even be defined unless $k = m$.)

The proof compares the (j, i) entries of both sides. It uses the *row-column rule* (9/8 notes, p. 54), which the notes recall: the (i, j) entry of a product CD is row i of C times column j of D ,

$$\begin{bmatrix} c_{i1} & \cdots & c_{in} \end{bmatrix} \begin{bmatrix} d_{1j} \\ \vdots \\ d_{nj} \end{bmatrix} = c_{i1}d_{1j} + \cdots + c_{in}d_{nj}.$$

Proof of $(AB)^T = B^T A^T$ (blanks filled in)

Left side. By the definition of the transpose and the row-column rule,

$$(j, i) \text{ entry of } (AB)^T = (i, j) \text{ entry of } AB = a_{i1}b_{1j} + \cdots + a_{in}b_{nj}. \quad (*)$$

Right side. By the row-column rule, the (j, i) entry of $B^T A^T$ is (row j of B^T) times (column i of A^T). Row j of B^T is column j of B (laid flat), and column i of A^T is row i of A (stood up). So

$$\begin{aligned} (j, i) \text{ entry of } B^T A^T &= [\text{column } j \text{ of } B]^T [\text{row } i \text{ of } A]^T = \begin{bmatrix} b_{1j} & \cdots & b_{nj} \end{bmatrix} \begin{bmatrix} a_{i1} \\ \vdots \\ a_{in} \end{bmatrix} \\ &= b_{1j}a_{i1} + \cdots + b_{nj}a_{in}. \end{aligned}$$

Compare. Multiplication of numbers is commutative, so $b_{1j}a_{i1} + \cdots + b_{nj}a_{in} = a_{i1}b_{1j} + \cdots + a_{in}b_{nj}$, which is $(*)$. (The arrow under the first “=” on p. 61 points from this final expression back to $(*)$.) All entries agree, so $(AB)^T = B^T A^T$. ■

Check on an example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 0 \\ -1 & 2 \end{bmatrix}$ give $AB = \begin{bmatrix} 3 & 4 \\ 11 & 8 \end{bmatrix}$ (9/8 notes, p. 53), and $B^T A^T = \begin{bmatrix} 5 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 11 \\ 4 & 8 \end{bmatrix} = (AB)^T$. ✓

Ex (p. 62): $(ABCD)^T = D^T C^T B^T A^T$. Why?

Apply the theorem repeatedly, peeling off one factor at a time (products can be grouped freely by associativity, 9/8 notes, p. 52):

$$\begin{aligned} (ABCD)^T &= ((ABC)D)^T = D^T (ABC)^T = D^T ((AB)C)^T \\ &= D^T C^T (AB)^T = D^T C^T B^T A^T. \end{aligned}$$

Meaning: the transpose of a product is the product of the transposes *in reverse order*. (Checked numerically on random 3×3 matrices.) ✓

2 The inverse of a matrix (pp. 62–65)

An $n \times n$ matrix A is invertible when some matrix C undoes it from both sides; that C is unique, so we may call it A^{-1} .

Definition (invertible matrix, p. 62)

An $n \times n$ matrix A is **invertible** (or **nonsingular**) if there is an $n \times n$ matrix C with

$$CA = I_n = AC.$$

We denote C by A^{-1} .

The notation A^{-1} “makes sense because” there is only one such C :

Theorem (p. 62)

A^{-1} is determined by A : an invertible matrix has exactly one inverse.

Proof (pp. 62–63). Suppose C and B are both inverses of A . Compute the product CAB in two ways (associativity, 9/8 notes, p. 52):

$$C\underline{AB} = CI = C \quad \text{and} \quad \underline{CA}B = IB = B.$$

The first uses $AB = I$ (B is an inverse); the second uses $CA = I$ (C is an inverse). Both compute the same matrix, so $C = B$. ■

Example (p. 63): the inverse of a 2×2 matrix**Fact (p. 63)**

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible $\iff ad - bc \neq 0$, and then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The recipe: swap the diagonal entries a, d , negate the off-diagonal entries b, c , and divide by the number $ad - bc$.

Why? Because both products equal $(ad - bc)I$

The notes multiply out (the entries $ab - ba, cd - dc$ vanish):

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & -ab + ba \\ cd - dc & -cb + da \end{bmatrix} = (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and “similarly”

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} da - bc & db - bd \\ -ca + ac & -cb + ad \end{bmatrix} = (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

If $ad - bc \neq 0$, divide both equations by $ad - bc$: the matrix $C = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ satisfies $AC = I = CA$. By the definition, A is invertible and $C = A^{-1}$. (Both products were verified symbolically.) ✓

The other direction (an addition): if $ad - bc = 0$, then A is not invertible. The first product above becomes $A \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = 0$. If A had an inverse, multiplying on the left by A^{-1} would give $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1}0 = 0$, so $a = b = c = d = 0$ and A is the zero matrix. But $0 \cdot C = 0 \neq I$ for every C , so the zero matrix is not invertible. This contradiction shows A is not invertible.

Fill-in (p. 64): $\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}^{-1} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$

Here $a = 2$, $b = 5$, $c = -3$, $d = -7$, so $ad - bc = 2(-7) - 5(-3) = -14 + 15 = 1$. The formula gives

$$\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}^{-1} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} \cdot \frac{1}{-14 + 15} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}.$$

Check: $\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -14+15 & -10+10 \\ 21-21 & 15-14 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \checkmark$

Rules for inverses (pp. 64–65)

Theorem (p. 64)

If A, B are invertible $n \times n$ matrices, then

1. A^{-1} is invertible and $(A^{-1})^{-1} = A$;
2. AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$;
3. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

Proof (“plug in to check”, pp. 64–65). In each part we have a candidate C for the inverse; we plug it into the definition ($CX = I = XC$) and check. By the uniqueness theorem (p. 62), passing the check means C is the inverse.

1. $A^{-1}A = I = AA^{-1}$. Read this with $X = A^{-1}$ and $C = A$: it says $CX = I = XC$. So A is the inverse of A^{-1} .

2. Using associativity,

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I,$$

and similarly $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$.

3. Using $(AB)^T = B^T A^T$ (p. 61) and $I^T = I$:

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I,$$

and similarly $(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$. ■

Ex (p. 65): $(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$. Why?

(Here A, B, C, D are invertible $n \times n$ matrices.) Use part 2 of the theorem three times, exactly as for the transpose:

$$\begin{aligned} (ABCD)^{-1} &= ((ABC)D)^{-1} = D^{-1}(ABC)^{-1} = D^{-1}((AB)C)^{-1} \\ &= D^{-1}C^{-1}(AB)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}. \end{aligned}$$

Check by plugging in: $(ABCD)(D^{-1}C^{-1}B^{-1}A^{-1})$ collapses from the middle: $DD^{-1} = I$, then

$CC^{-1} = I$, then $BB^{-1} = I$, then $AA^{-1} = I$. ✓ (Also verified numerically on random 3×3 matrices.)

Why it makes sense: to undo “first D , then C , then B , then A ” you undo A first and D last, like taking off socks and shoes.

Ex (p. 65). For invertible P and A , part 2 (applied to three factors) and part 1 give

$$(PAP^{-1})^{-1} = (P^{-1})^{-1}A^{-1}P^{-1} = PA^{-1}P^{-1}.$$

Check: $(PAP^{-1})(PA^{-1}P^{-1}) = PA(P^{-1}P)A^{-1}P^{-1} = P(AA^{-1})P^{-1} = PP^{-1} = I$. ✓

Ex (p. 65): powers. $(A^3)^{-1} = (AAA)^{-1} = A^{-1}A^{-1}A^{-1} = (A^{-1})^3$. This common value is written A^{-3} . Together with the definition $A^0 = I$ (boxed on p. 65), the powers A^k now make sense for every integer k when A is invertible: $A^{-k} = (A^{-1})^k$.

3 Computing A^{-1} in practice (pp. 66–67)

For larger matrices there is no simple formula; instead, row reduce $[A \ I_n]$, and if the left half becomes I_n , the right half is A^{-1} .

Theorem (p. 66)

An $n \times n$ matrix A is invertible $\iff$ RREF $A = I_n$.

Theorem (p. 66)

If A is an invertible $n \times n$ matrix, then row reduction of the $n \times 2n$ matrix $[A \ I_n]$ gives $[I_n \ A^{-1}]$.

The notes say that “the proof of the above theorems takes work and we shall finish it later”. It is finished in this same lecture: see Sections 4–5, and the summary of the logic at the end of Section 5.

Example (p. 66): compute the inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

Put I_3 next to A and row reduce. Step 1 swaps rows 1 and 2 (there is a 0 in the top-left corner). Step 2 clears column 3 above the pivot in row 3, with $R_1 - 3R_3$ and $R_2 - 2R_3$:

$$[A \ I] = \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow[R_2 - 2R_3]{R_1 - 3R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right].$$

The left half is I_3 , so by the theorem

$$A^{-1} = \begin{bmatrix} 0 & 1 & -3 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Fill-in (p. 66): the check AA^{-1} equals I_3

The notes start the check and stop at the third column:

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -3 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2+2 \\ 0 & 1 & -3+3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

For example, the $(1,3)$ entry is row 1 of A times column 3 of A^{-1} : $0(-3) + 1(-2) + 2(1) = -2 + 2 = 0$.

Check the other side too (the definition needs both): $A^{-1}A = \begin{bmatrix} 0 & 1 & -3 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3-3 \\ 0 & 1 & 2-2 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$

✓

Ex (p. 67): solving $Ax = b$ when A^{-1} is known

If we are given A^{-1} , multiply $Ax = b$ on the left by A^{-1} :

$$A^{-1}Ax = A^{-1}b, \quad \text{and} \quad A^{-1}Ax = Ix = x, \quad \text{so} \quad x = A^{-1}b.$$

“But it is faster to simply solve $Ax = b$ by row reduction.”

Why is row reducing $[A \ b]$ faster than computing $A^{-1}b$? Because finding A^{-1} is n systems’ worth of work

The notes give the reason: “computing A^{-1} involves solving n linear systems of the form $Ax = e_i$ ” (one for each column e_i of I_n ; this is explained on pp. 68–70, Section 4).

- Row reducing $[A \ b]$ carries **one** extra column along through the row operations.
- Row reducing $[A \ I_n]$ carries n extra columns along, and afterwards we still have to multiply A^{-1} by b (another n^2 multiplications).

The work on the left half A is the same in both, so the inverse route does strictly more arithmetic. For large n it costs roughly three times as much (this ratio is a standard operation count, not from your notes).

Check on the system of p. 69: row reduction gives $x = (-1, 1)$ directly. The inverse route first needs $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}^{-1} = \frac{1}{2-6} \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 \\ 3/4 & -1/4 \end{bmatrix}$, then $A^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1/2-1/2 \\ 3/4+1/4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$: the same answer, with more work. ✓

4 Invertible means n pivots; why the algorithm works (pp. 67–70)

An invertible $n \times n$ matrix has n pivots, and then row reducing $[A \ I_n]$ solves the n systems $Ax = e_1, \dots, Ax = e_n$ at once; their solutions are the columns of A^{-1} .

Theorem (p. 67)

Let A be a square ($n \times n$) matrix. Then $1 \Rightarrow 2 \iff 3$, where

1. A is invertible;
2. $Ax = b$ is consistent for each b in $\mathbb{R}^n$;
3. $Ax = 0$ has only the zero solution.

Proof (p. 68). Rephrase 2 and 3 in terms of pivots, using two facts from the 9/3 notes:

- $2 \iff A$ has a pivot in each row (9/3 notes, p. 45; Lay §1.4, Theorem 4);
- $3 \iff A$ has a pivot in each column (9/3 notes, p. 46: no free variables).

$2 \iff 3$ **since A is square.** Each pivot sits in its own row and its own column. So “a pivot in each of the n rows” and “a pivot in each of the n columns” both say the same thing: A has n pivots.

$1 \Rightarrow 3$. Suppose A is invertible and $Ax = 0$. Multiply on the left by A^{-1} :

$$x = Ix = A^{-1}Ax = A^{-1}0 = 0.$$

So the only solution is $x = 0$.

$1 \Rightarrow 2$ then follows from $1 \Rightarrow 3 \Rightarrow 2$. (A direct proof is on p. 73, Section 5.) ■

“Thus we showed: A is invertible $\Rightarrow A$ has n pivots.” Since A is square, n pivots means a pivot in every row and column, so $\text{RREF } A = I_n$. This is the “ $\Rightarrow$ ” half of the first theorem on p. 66.

Why does the algorithm for finding A^{-1} work when A is invertible? It solves $Ax = e_i$ for all i at once

The notes’ argument (pp. 68–69), step by step:

1. We know A has n pivots (just shown). So $\text{RREF } A = I_n$, and $[A \ I_n]$ row reduces to $[\text{RREF } A \ B] = [I_n \ B]$ for some $n \times n$ matrix B . The columns of I_n here are $e_1, \dots, e_n$.
2. There are no pivots in the B -columns (all n pivots are already in the left half). Each row operation acts on each column separately: what happens to one column does not depend on the other columns. So the B -columns “change independently of each other via row reduction”.
3. Hence the algorithm “simultaneously solves n linear systems” $Ax = e_1, \dots, Ax = e_n$: the same row operations that turn $[A \ e_i]$ into $[I_n \ d_i]$ turn $[A \ e_1 \ \dots \ e_n]$ into $[I_n \ d_1 \ \dots \ d_n]$.
4. “The solution of $Ax = e_i$ is $x = A^{-1}e_i$ ” (multiply by A^{-1} as on p. 67), and $A^{-1}e_i$ is the i -th column of A^{-1} (9/3 notes, p. 41: Ae_i is column i of A , applied to the matrix A^{-1}).

So the right half $B = [d_1 \ \dots \ d_n]$ is $[A^{-1}e_1 \ \dots \ A^{-1}e_n] = A^{-1}$. This is written out as the theorem on p. 70 below.

The step “[$A \ e_i$] becomes [$I_n \ d_i$], so d_i is the solution” uses a fact recalled on p. 69:

Recall (p. 69)

If A has n pivots, then $[A \mid b]$ row reduces to $[I_n \mid d]$, and $Ad = b$, because $Ax = b \iff I_n x = d$ (row operations do not change the solution set), and $I_n x = d$ means $x = d$.

Ex (p. 69): solve $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Row reduce the augmented matrix: clear below the first pivot ($R_2 - 3R_1$), scale row 2 by $-\frac{1}{4}$, then clear above the second pivot ($R_1 - 2R_2$):

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 2 & -1 \end{array} \right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -4 & -4 \end{array} \right] \xrightarrow{R_2 / (-4)} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \end{array} \right] \xrightarrow{R_1 - 2R_2} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array} \right].$$

The last matrix says $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, so $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Check: $1(-1) + 2(1) = 1$ and $3(-1) + 2(1) = -1$. ✓

Theorem (p. 70)

If A is an invertible $n \times n$ matrix, then $[A \mid I_n]$ row reduces to $[I_n \mid A^{-1}]$.

Proof (p. 70). As we saw above, $[A \mid e_1 \cdots e_n]$ row reduces to $[I_n \mid d_1 \cdots d_n]$, and (by the same row operations) $[A \mid e_i]$ row reduces to $[I_n \mid d_i]$. So

$$Ax = e_i \iff I_n x = d_i.$$

Thus $d_i = x = A^{-1}e_i$ is the i -th column of A^{-1} . Since this holds for every i , the right half is A^{-1} . ■

What remains to show: an $n \times n$ matrix with n pivots is invertible (the “ $\Leftarrow$ ” half of the first theorem on p. 66). “For this we need a new concept.”

5 Elementary matrices; n pivots implies invertible (pp. 70–73)

Each row operation is left multiplication by an elementary matrix, and each elementary matrix is invertible; so reducing A to I_n writes A^{-1} as a product of elementary matrices.

Definition (elementary matrix, p. 70)

An **elementary matrix** is a matrix obtained from I by a single row operation.

Ex (p. 71): elementary 2×2 matrices. Here c is a scalar (with $c \neq 0$ for the two scalings, since multiplying a row by 0 is not a row operation):

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$	$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$
swap rows	scale row 2 by c	scale row 1 by c	add $c \cdot (\text{row } 1)$ to row 2	add $c \cdot (\text{row } 2)$ to row 1

Each one is what you get by doing that row operation to I_2 .

Elementary matrices encode row operations (p. 71). Multiplying a matrix on the left by an elementary matrix E performs the same row operation on it. With $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$:

$$\begin{aligned} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} &= \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \text{ (rows swapped),} & \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ 3c & 4c \end{bmatrix} \text{ (row 2 scaled),} \\ \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} &= \begin{bmatrix} c & 2c \\ 3 & 4 \end{bmatrix} \text{ (row 1 scaled),} & \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ c \cdot 1 + 3 & 2c + 4 \end{bmatrix} \text{ (} R_2 + cR_1 \text{).} \end{aligned}$$

(All four products verified symbolically.) ✓ *Why it works:* by the row-column rule, row i of EM is (row i of E) times M , a combination of the rows of M with coefficients from row i of E . For instance row 2 of $\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$ is $(c, 1)$, so row 2 of the product is $c \cdot (\text{row } 1) + 1 \cdot (\text{row } 2)$.

Why are elementary matrices invertible? Every row operation can be undone by another row operation

The notes' hint: if E_1 does a row operation and E_2 does the reverse operation, then $E_2E_1A = A$ for every A , so $E_2E_1 = I$.

In detail: each row operation has a reverse operation of the same type:

- swap rows i and $j \leftrightarrow$ swap them back;
- scale row i by $c \neq 0 \leftrightarrow$ scale row i by $1/c$ (this is where $c \neq 0$ is needed);
- add $c \cdot R_j$ to $R_i \leftrightarrow$ add $-c \cdot R_j$ to R_i .

Let E_1 be the elementary matrix of an operation and E_2 that of its reverse. Doing the operation and then its reverse changes nothing, so $E_2E_1A = A$ for every A . Taking $A = I$ gives $E_2E_1 = I$. The same reasoning with the two operations in the other order gives $E_1E_2 = I$. So E_1 is invertible with $E_1^{-1} = E_2$, which is again elementary.

Check: $\begin{bmatrix} 1 & 0 \\ -c & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -c+c & 1 \end{bmatrix} = I$ and $\begin{bmatrix} 1/c & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} = I$. ✓

Ex (p. 72): which E encodes $R_2 + 2R_1$? $E = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The row operation in the notes is “add 2 times row 1 to row 2” (marked +2 ①):

$$\begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2+2R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

To find E , do this operation to I_3 (the definition): row 2 of I_3 , which is $(0, 1, 0)$, becomes $(0, 1, 0) + 2(1, 0, 0) = (2, 1, 0)$. That gives the matrix in the title, as in the notes.

Check:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2-2 & 2+1 & 2+0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}. \checkmark$$

Theorem (p. 72)

If an $n \times n$ matrix A has n pivots, then A is invertible, and $A^{-1} = E_k \cdots E_1$, where $E_1, \dots, E_k$ are the elementary matrices of the row operations that reduce A to I_n .

Proof (pp. 72–73). Since A is square with n pivots, $\text{RREF } A = I_n$. Realize each row operation going from A to I_n by left multiplication by an elementary matrix: the first operation is E_1 , the second E_2 , and so on, so

$$E_k \cdots E_2 E_1 A = I_n.$$

Each E_i is invertible (p. 71). Multiply both sides on the left by E_k^{-1} , then by $E_{k-1}^{-1}, \dots$, then by E_1^{-1} :

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I = E_1^{-1} \cdots E_k^{-1}.$$

So A is invertible as a product of invertible matrices (p. 64, part 2, applied repeatedly), and by p. 64, parts 1–2,

$$A^{-1} = (E_1^{-1} \cdots E_k^{-1})^{-1} = (E_k^{-1})^{-1} \cdots (E_1^{-1})^{-1} = E_k \cdots E_1. \quad \blacksquare$$

Worked example (an addition). The row reduction on p. 69 used three operations on $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$. Their elementary matrices (each is that operation done to I_2) are

$$E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} (R_2 - 3R_1), \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{4} \end{bmatrix} (R_2 / (-4)), \quad E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} (R_1 - 2R_2).$$

Then $E_1 A = \begin{bmatrix} 1 & 2 \\ 0 & -4 \end{bmatrix}$, $E_2 E_1 A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $E_3 E_2 E_1 A = I_2$, and

$$A^{-1} = E_3 E_2 E_1 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix},$$

which matches the 2×2 formula: $\frac{1}{2-6} \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}$. $\checkmark$

The full logic of pp. 66–73. Combining the pieces proves both theorems of p. 66:

- A invertible $\Rightarrow n$ pivots $\Rightarrow \text{RREF } A = I_n$ (p. 68);
- $\text{RREF } A = I_n \Rightarrow n$ pivots $\Rightarrow A$ invertible (pp. 72–73);
- and then $[A \ I_n] \rightarrow [I_n \ A^{-1}]$ (p. 70).

The theorem on p. 72 also gives a second reason the algorithm works: the operations $E_1, \dots, E_k$ applied to $[A \ I_n]$ give $[E_k \cdots E_1 A \ E_k \cdots E_1 I] = [I_n \ A^{-1}]$ (Lay §2.2). It also gives the converse $3 \Rightarrow 1$ of the theorem on p. 67: $Ax = 0$ having only the zero solution means n pivots, hence invertible. So for square matrices, 1, 2 and 3 are all equivalent. The 9/15 notes collect this and more into the Invertible Matrix Theorem (Lay §2.3).

Ex (p. 73): A invertible $\Rightarrow Ax = b$ is consistent for each b , with the unique solution $x = A^{-1}b$

The notes write $x = A^{-1}Ax = A^{-1}b$. This line shows: if x is a solution, then $x = A^{-1}b$, so there is at most one solution. For consistency we also need that $A^{-1}b$ really is a solution. Plug it in:

$$A(A^{-1}b) = (AA^{-1})b = Ib = b.$$

So $Ax = b$ has a solution for every b (this is $1 \Rightarrow 2$ of p. 67, proved directly), and it is exactly one solution.

Check with p. 69: $A^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, which solves the system there. ✓

6 Left and right inverses; the columns Ae_i (pp. 73–74)

For a non-square matrix, an inverse from one side need not come with an inverse from the other side, and one-sided inverses need not be unique.

Definition (left and right inverses, p. 73)

C is a **left inverse** of A if $CA = I$.

C is a **right inverse** of A if $AC = I$.

So A is invertible exactly when it is square and has a matrix C that is both a left and a right inverse.

Ex (p. 74): $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has infinitely many left inverses $C = [1 \ x]$, but no right inverse

A is 2×1 , so for CA to be defined and equal to $I_1 = [1]$, C must be 1×2 . Every $C = [1 \ x]$ works, whatever the number x :

$$\begin{bmatrix} 1 & x \\ 1 \times 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \times 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \times 1 \end{bmatrix} = I_1.$$

(More precisely, $C = [p \ q]$ is a left inverse $\iff p = 1$.) So a left inverse is *not* unique: the

uniqueness proof of p. 62 needs inverses from both sides.

In the other order, the notes compute

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{2 \times 1} \begin{bmatrix} 1 & 0 \end{bmatrix}_{1 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{2 \times 2} \neq I_2.$$

In fact *no* C is a right inverse: for any $C = [p \ q]$, $AC = \begin{bmatrix} p & q \\ 0 & 0 \end{bmatrix}$ has bottom row zero, but the bottom row of I_2 is $(0, 1)$. So A has left inverses but no right inverse, and it is not invertible (it is not even square).

Check: with $x = 5$, $[1 \ 5] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 + 0 = 1$. ✓

Looking ahead. For *square* matrices this cannot happen: if an $n \times n$ matrix has a left inverse or a right inverse, it is invertible, and that one-sided inverse is A^{-1} . This is part of the Invertible Matrix Theorem (9/15 notes; Lay §2.3). For example, if $CA = I$, then $Ax = 0$ gives $x = CAx = C0 = 0$, so A is invertible by the logic box above.

The columns of A (p. 74). The notes end by recalling

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{row } i = \text{column } i \text{ of } I_n,$$

and

$$Ae_i = i\text{-th column of } A = T(e_i), \quad \text{where } T(x) = Ax.$$

Why: $Ae_i = 0 \cdot a_1 + \cdots + 1 \cdot a_i + \cdots + 0 \cdot a_n = a_i$, where $a_1, \dots, a_n$ are the columns of A (9/3 notes, p. 41). Read through the linear map $T(x) = Ax$, this says that column i of the standard matrix is $T(e_i)$ (9/3 notes, p. 41; 9/8 notes, p. 49). It is the fact used on p. 69: $A^{-1}e_i$ is the i -th column of A^{-1} .

Check with p. 66: $A^{-1}e_3 = \begin{bmatrix} 0 & 1 & -3 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$, the third column. ✓

Summary

Fact	Formula
Transpose of an $m \times n$ matrix	$n \times m$; (j, i) entry of $A^T = a_{ij}$
Transpose rules	$(A^T)^T = A$, $(A + B)^T = A^T + B^T$, $(rA)^T = rA^T$
Transpose of a product	$(AB)^T = B^T A^T$
Symmetric matrix	$A^T = A$; at most $n(n+1)/2$ different entries
Inverse	$CA = I_n = AC$; unique, written A^{-1}
2×2 inverse	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ if $ad-bc \neq 0$
Inverse rules	$(A^{-1})^{-1} = A$, $(AB)^{-1} = B^{-1}A^{-1}$, $(A^T)^{-1} = (A^{-1})^T$
Algorithm	$[A \ I_n] \rightarrow [I_n \ A^{-1}]$
Invertible (square A)	$\iff n$ pivots $\iff \text{RREF } A = I_n$
Solving with A^{-1}	$x = A^{-1}b$ (but row reducing $[A \ b]$ is faster)
Elementary matrix	one row operation applied to I ; EM = that operation on M
Inverse via elementary matrices	$E_k \cdots E_1 A = I \Rightarrow A^{-1} = E_k \cdots E_1$
Left / right inverse	$CA = I$ / $AC = I$; may exist alone for non-square A

Corrections to the handwritten notes

Page	As written	Should be	Why
65	$(PAP^{-1})^{-1} = PA^{-1}P^{-1}$, no hypothesis on A	add “ A invertible”	A^{-1} must exist (only P is visibly invertible)
71	“scale row 2”, “scale row 1” (by c)	by $c \neq 0$	scaling by 0 is not a row operation, and the answer on p. 71 needs $1/c$
71	“add 1st row to row 2”, “add 2nd row to 1st row”	add $c \cdot$ (1st row) to row 2; add $c \cdot$ (2nd row) to row 1	the matrices have entry c , not 1
72	$E = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$E = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $E \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the result	E is the first factor alone, not the product