

Linear Algebra Notes — Lecture 9/17

Bases of $\text{Nul } A$ and $\text{Col } A$, Coordinates, Dimension and Rank

Lecture pages 90–104 · Lay §2.8–2.9, §4.1

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Overview

Last time (9/15 notes, pp. 84–89) we met subspaces of $\mathbb{R}^n$, the null space $\text{Nul } A$, the column space $\text{Col } A$, and the definition of a *basis*: an independent set of vectors in S that spans S . This lecture answers the natural follow-up questions:

1. How do we actually **find** a basis of $\text{Nul } A$ and of $\text{Col } A$? (pp. 90–91)
2. Once we have a basis, every vector gets **coordinates**. (pp. 92–94)
3. All bases of a subspace have the same size, the **dimension**. This gives the **rank** and the **Rank Theorem** $\dim \text{Col } A + \dim \text{Nul } A = n$. (pp. 95–99)
4. If $\dim S = k$, then k vectors in S form a basis as soon as they are independent *or* span. Every nonzero subspace has a basis. (pp. 100–103)

5. A first look at abstract **vector spaces**. (p. 104)

The professor says that everything up to Lay §2.9 is on Midterm 1.

Every blank and question in the notes is answered in a blue box. Typos are listed at the end.

Notation used throughout.

- $\text{Nul } A = \{x : Ax = 0\}$, $\text{Col } A = \text{span}\{\text{columns of } A\}$ (9/15 notes, pp. 84–85).
- $\text{Ker } T = \text{kernel of a linear map } T = \text{all } x \text{ with } T(x) = 0$; if $T(x) = Ax$ then $\text{Ker } T = \text{Nul } A$.
- $[x]_\beta = \text{the coordinate vector of } x \text{ relative to a basis } \beta$ (p. 93).
- $\dim S = \text{dimension of } S$ (p. 95); $\text{rank } A = \text{number of pivots of } A$ (p. 97).
- $e_1, \dots, e_n = \text{the standard basis of } \mathbb{R}^n$ (e_i has a 1 in slot i , zeros elsewhere).

1 Bases of $\text{Nul } A$ and $\text{Col } A$ (pp. 90–91)

Solving $Ax = 0$ in parametric form gives a basis of $\text{Nul } A$; the pivot columns of A itself give a basis of $\text{Col } A$.

Theorem (p. 90)

The usual algorithm for solving $Ax = 0$ (row reduce, solve for the pivot variables, write the solution in parametric vector form) gives a basis of $\text{Nul } A$: one vector for each free variable.

The notes give a “proof” by the example on p. 89 of the 9/15 notes. Here is the general argument.

Proof (filled in): the parametric vectors form a basis of $\text{Nul } A$

Suppose the free variables are $x_{j_1}, \dots, x_{j_p}$. The algorithm writes every solution as

$$x = x_{j_1} u_1 + \dots + x_{j_p} u_p.$$

Span. Every solution has this form, so $u_1, \dots, u_p$ span $\text{Nul } A$.

Independent. Look at slot j_1 : the vector u_1 has a 1 there (setting $x_{j_1} = 1$ and the other free variables to 0), and $u_2, \dots, u_p$ have a 0 there. The same is true for each slot j_i . So if $c_1 u_1 + \dots + c_p u_p = 0$, then reading slot j_i gives $c_i = 0$ for each i .

Illustration (from p. 91 below): $\text{Nul } A$ has basis $u_1 = (2, -1, 1, 0)$, $u_2 = (-3, 2, 0, 1)$ with free variables x_3, x_4 . Slots 3–4 of u_1, u_2 are $(1, 0)$ and $(0, 1)$, so independence is visible at a glance.

✓

Theorem (p. 90; Lay §2.8)

The pivotal columns of A form a basis of $\text{Col } A$.

“Proof” in the notes. We saw before (9/15 notes) that the pivotal columns are independent. The notes then ask why they span.

Why do the pivotal columns span Col A ? Every other column is a combination of them

Let R be the RREF of A . Row operations do not change the solutions of $Ax = 0$, so

$$Ax = 0 \iff Rx = 0.$$

A solution x is exactly a list of weights giving a dependence $x_1a_1 + \cdots + x_na_n = 0$ among the columns. So **the columns of A satisfy exactly the same dependence relations as the columns of R .**

In R the pivotal columns are $e_1, \dots, e_r$, and each nonpivotal column is obviously a combination of them (its entries are the weights). That relation carries over to A . Hence each nonpivotal column of A lies in the span of the pivotal columns of A , and any combination of *all* columns can be rewritten using only the pivotal ones. So the pivotal columns span Col A . The example on p. 91 shows this concretely.

Example (p. 91)

Take

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & -1 & 0 \\ 2 & 3 & -1 & 0 \end{bmatrix} = [a_1 \ a_2 \ a_3 \ a_4].$$

(The notes write the $(3,3)$ entry as 1; it must be -1 for the row reduction shown to end in a zero row. See the corrections table.) Subtract $2 \times$ row 1 from rows 2 and 3, then subtract the new row 2 from row 3:

$$A \rightarrow \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow R = \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = [e_1 \ e_2 \ r_3 \ r_4]$$

(the last step is row 1 $-$ row 2). The columns of R visibly satisfy

$$r_3 = -2e_1 + e_2, \quad r_4 = 3e_1 - 2e_2,$$

that is,

$$2e_1 - e_2 + r_3 + 0r_4 = 0, \quad -3e_1 + 2e_2 + 0r_3 + 1 \cdot r_4 = 0.$$

So $(2, -1, 1, 0)$ and $(-3, 2, 0, 1)$ are solutions of $Rx = 0$, hence of $Ax = 0$. Read as relations among the columns of A :

$$2a_1 - a_2 + a_3 = 0, \quad -3a_1 + 2a_2 + a_4 = 0.$$

So a_3, a_4 are in $\text{span}\{a_1, a_2\}$, and the pivotal columns $\{a_1, a_2\}$ form a basis of Col A .

Check: $2(1, 2, 2) - (1, 3, 3) + (-1, -1, -1) = (0, 0, 0)$ and $-3(1, 2, 2) + 2(1, 3, 3) + (1, 0, 0) = (0, 0, 0)$.
✓

Use the columns of A , not of R . The pivotal columns of R (e_1, e_2) usually do *not* lie in Col A . Here every column of A has equal 2nd and 3rd entries, but $e_1 = (1, 0, 0)$ does too while

$e_2 = (0, 1, 0)$ does not. Row reduction tells you *which* columns to take; you then take them from A .

2 More on bases; coordinates (pp. 92–94)

A nonzero subspace has infinitely many bases; any one of them lays a “grid” on the subspace, and the grid position of a vector is its coordinate vector.

What is the basis of $\{0\}$? It has no basis (of nonzero vectors)

The notes: “No basis.” Any vector in $\{0\}$ is 0, and any set containing 0 is dependent ($1 \cdot 0 = 0$ is a nontrivial relation). So no nonempty set of vectors in $\{0\}$ is independent.

Beside this the professor boxed a remark. My reading of it is “ $\emptyset = 0$ ”: the usual convention is that the *empty set* counts as the basis of $\{0\}$ (the span of no vectors is $\{0\}$). This convention is what makes $\dim\{0\} = 0$ on p. 95 consistent with “dimension = number of vectors in a basis”.

How many bases does a nonzero subspace S have? Infinitely many

Take one basis $\{v_1, \dots, v_k\}$ (one exists by the theorem on p. 103). The notes’ example: $\{2v_1, v_2, \dots, v_k\}$ is also a basis. More generally $\{cv_1, v_2, \dots, v_k\}$ is a basis for every $c \neq 0$:

- *Span*: $x = a_1v_1 + \dots + a_kv_k = \frac{a_1}{c}(cv_1) + a_2v_2 + \dots + a_kv_k$.
- *Independent*: $d_1(cv_1) + d_2v_2 + \dots + d_kv_k = 0$ is a relation among $v_1, \dots, v_k$ with weights $(cd_1, d_2, \dots, d_k)$, so all of these are 0, and $c \neq 0$ gives $d_1 = 0$.

Different c give different sets (since $v_1 \neq 0$), so there are infinitely many bases.

Example (p. 92): a basis as a grid

A basis $\{v_1, v_2\}$ of $\mathbb{R}^2$ draws a grid: lines parallel to v_2 through the multiples of v_1 , and lines parallel to v_1 through the multiples of v_2 . The point $c_1v_1 + c_2v_2$ is found by walking c_1 steps along v_1 and c_2 steps along v_2 . Figure 1 shows the standard grid (left) and a skewed grid (right; the vectors $v_1 = (0.7, -0.35)$, $v_2 = (1.2, 0.5)$ are my choice to match the sketch).

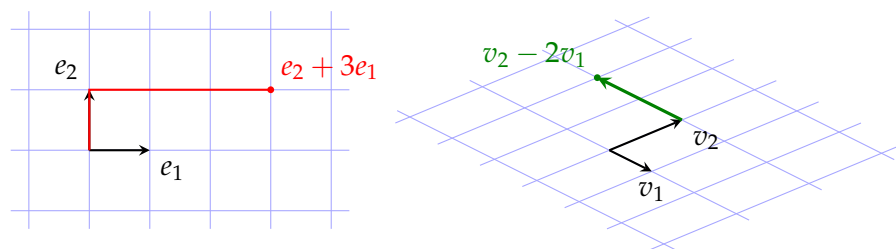


Figure 1: Left: the standard grid; $e_2 + 3e_1$ is reached by one step up and three steps right. Right: the grid of a basis $\{v_1, v_2\}$; $v_2 - 2v_1$ is reached from v_2 by two steps backwards along v_1 (green).

Sketch $v_2 - 2v_1$: its grid coordinates relative to $\{v_1, v_2\}$ are $(-2, 1)$

Start at 0, take one step along v_2 , then two steps *against* v_1 (green arrow). So $v_2 - 2v_1 = (-2)v_1 + 1 \cdot v_2$, and the notes record the coordinate vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Likewise $e_2 + 3e_1$ has coordinates $(3, 1)$ in the standard grid.

Check (with my v_1, v_2): $v_2 - 2v_1 = (1.2 - 1.4, 0.5 + 0.7) = (-0.2, 1.2)$, the green point. ✓

Definition (p. 93): coordinates

Let $\beta = \{b_1, \dots, b_k\}$ be a basis of a subspace S . For x in S , its **coordinates relative to β** are the numbers $c_1, \dots, c_k$ with

$$x = c_1 b_1 + \dots + c_k b_k.$$

We write $[x]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$.

The definition only makes sense because the c_i exist (the b_i span S) and are unique (proved on p. 94).

Example (pp. 93–94)

Let $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Why is $\beta = \{v_1, v_2\}$ a basis of $\mathbb{R}^2$? Independent and spanning

Independence is clear: neither vector is a multiple of the other, and two vectors are dependent only if one is a multiple of the other.

Span: the notes write the standard basis in terms of v_1, v_2 :

$$e_1 = \frac{v_1 + v_2}{2}, \quad e_2 = \frac{v_1 - v_2}{2}.$$

Every $x = x_1 e_1 + x_2 e_2$ is a combination of e_1, e_2 , hence of v_1, v_2 . So $\text{span}\{v_1, v_2\} = \mathbb{R}^2$.

Check: $\frac{1}{2}((1, 1) + (1, -1)) = (1, 0)$ and $\frac{1}{2}((1, 1) - (1, -1)) = (0, 1)$. ✓

Find $[e_1]_\beta$: it is $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$

We need c_1, c_2 with $e_1 = c_1 v_1 + c_2 v_2$, that is, a matrix equation whose coefficient matrix has columns v_1, v_2 :

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

By the 2×2 inverse formula $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (Lay §2.2), with $ad - bc = -1 - 1 = -2$:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}.$$

This matches $e_1 = \frac{1}{2}v_1 + \frac{1}{2}v_2$ found above.

Check: $\frac{1}{2}(1, 1) + \frac{1}{2}(1, -1) = (1, 0) = e_1$. ✓

Theorem (p. 94): the notation $[x]_\beta$ makes sense

x and β determine $[x]_\beta$: the coordinates of x relative to a basis are unique.

Proof, including “Are $(c_1, \dots, c_k) = (d_1, \dots, d_k)$?”: Yes

Suppose $\beta = \{b_1, \dots, b_k\}$ and

$$x = c_1b_1 + \dots + c_kb_k = d_1b_1 + \dots + d_kb_k$$

for scalars $c_1, \dots, c_k, d_1, \dots, d_k$. Subtracting,

$$(c_1 - d_1)b_1 + \dots + (c_k - d_k)b_k = 0.$$

Since β is independent, every weight is 0: $c_i - d_i = 0$, so $c_i = d_i$ for each i . Hence the two coordinate vectors are equal. $\square$

Where the hypothesis is used: existence of the c_i uses that β spans; uniqueness uses that β is independent.

3 Dimension (pp. 95–97)

The dimension of a subspace is the size of any basis; this is well defined because all bases have the same size.

Definition (p. 95): dimension

The **dimension** of a subspace S of $\mathbb{R}^n$ is the number of vectors in a basis of S . It is written $\dim S$.

Examples (p. 95).

- $\dim \mathbb{R}^n = n$, since $\{e_1, \dots, e_n\}$ is a basis (9/15 notes, p. 88).
- $\dim\{0\} = 0$ (the empty basis; see p. 92 above).
- For an $m \times n$ matrix A : $\dim \text{Col } A = \#\text{pivots}$, since the pivotal columns are a basis (p. 90).
- $\dim \text{Nul } A = \#\text{free variables} = \#\text{nonpivotal columns}$, since the algorithm gives one basis vector per free variable (p. 90), and the free variables are exactly the variables of the nonpivotal

columns.

Example (p. 95): a hyperplane

Let $S = \{x \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\}$. Then $S = \text{Nul } A$ for the $1 \times n$ matrix $A = [1 \ 1 \ \cdots \ 1]$, because $Ax = x_1 + \cdots + x_n$.

$\dim S = n - 1$

$A = [1 \ 1 \ \cdots \ 1]$ is already in RREF with one pivot (column 1). Columns $2, \dots, n$ are nonpivotal, so $x_2, \dots, x_n$ are free and $\dim S = \dim \text{Nul } A = n - 1$.

A basis from the algorithm: $x_1 = -x_2 - \cdots - x_n$, so

$$x = x_2(-1, 1, 0, \dots, 0) + x_3(-1, 0, 1, 0, \dots, 0) + \cdots + x_n(-1, 0, \dots, 0, 1).$$

Check ($n = 3$): $(-1, 1, 0)$ and $(-1, 0, 1)$ both have entry sum 0, and they are independent (slots 2–3 are $(1, 0), (0, 1)$). ✓

Theorem (p. 96): dimension makes sense

Let S be a subspace of $\mathbb{R}^n$. Then any two bases of S have the same number of vectors.

Proof (p. 96). Let $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_\ell\}$ be bases of S . Arguing by contradiction, assume $k > \ell$. (If $k < \ell$, swap the roles of the two bases.) Since the b 's span S , write each

$$a_i = c_{i1}b_1 + \cdots + c_{i\ell}b_\ell = \underbrace{[b_1 \ \cdots \ b_\ell]}_B \underbrace{\begin{bmatrix} c_{i1} \\ \vdots \\ c_{i\ell} \end{bmatrix}}_{c_i} = Bc_i.$$

Here B is the matrix whose columns are the b 's, and c_i is the column of weights for a_i . Then

$$A = [a_1 \ \cdots \ a_k] = [Bc_1 \ \cdots \ Bc_k] = B[c_1 \ \cdots \ c_k] = BC,$$

using that B times a matrix is B times each column; C is the matrix with columns c_i .

What are the sizes of A, B, C ? $n \times k, n \times \ell, \ell \times k$

A has the k vectors $a_i \in \mathbb{R}^n$ as columns: $n \times k$. B has the ℓ vectors $b_j \in \mathbb{R}^n$ as columns: $n \times \ell$. Each c_i has ℓ entries and there are k of them: C is $\ell \times k$. (Consistent: $(n \times \ell)(\ell \times k) = n \times k$.)

“Since A has independent columns, so does C ” (the homework fact), and: is $k > \ell$ a contradiction? Yes

The homework fact (left as homework in the notes, p. 96; proved here for reference). If $Cx = 0$, then $Ax = B(Cx) = B0 = 0$. The columns of A are independent, so $x = 0$. Hence $Cx = 0$ only for $x = 0$: the columns of C are independent.

The contradiction. C is $\ell \times k$ with independent columns, so every one of its k columns is pivotal. Each pivot sits in a different row, and C has only ℓ rows, so $k \leq \ell$. (This is the lemma

on p. 102 below, or Lay §1.7: more vectors than entries means dependent.) But we assumed $k > \ell$. Contradiction, so $k \leq \ell$; by symmetry $\ell \leq k$, hence $k = \ell$. $\square$

Example (p. 97): a 5×7 matrix

Suppose A is a 5×7 matrix.

What can we say about $\dim \text{Col } A$? It is ≤ 5

$\dim \text{Col } A = \# \text{pivots}$, and there is at most one pivot per row; A has 5 rows. (Also: $\text{Col } A$ is a subspace of $\mathbb{R}^5$, and $\mathbb{R}^5$ holds at most 5 independent vectors.)

What can we say about $\dim \text{Nul } A$? It is ≥ 2

$\dim \text{Nul } A = \# \text{nonpivotal columns} = 7 - \# \text{pivots} \geq 7 - 5 = 2$. In particular $Ax = 0$ always has nonzero solutions.

4 Rank and the Rank Theorem (pp. 97–99)

The rank of A is $\dim \text{Col } A$, and rank plus nullity is the number of columns.

Definition (p. 97): rank

The **rank** of A is the number of pivots of A , written $\text{rank } A$. By p. 95, $\text{rank } A = \dim \text{Col } A$.

Ex: $\text{rank} \begin{bmatrix} 1 & 1 \\ 1 & 1 + \varepsilon \end{bmatrix} = 2$ if $\varepsilon \neq 0$, and $= 1$ if $\varepsilon = 0$

Row 2 – row 1 gives $\begin{bmatrix} 1 & 1 \\ 0 & \varepsilon \end{bmatrix}$. If $\varepsilon \neq 0$ there are two pivots; if $\varepsilon = 0$ the second row is zero and there is one pivot.

Meaning (the sketch in the notes): the columns $(1, 1)$ and $(1, 1 + \varepsilon)$ point in almost the same direction. For tiny $\varepsilon \neq 0$ they are still independent, so the rank jumps from 1 to 2 even though the matrix barely changed.

Rank is fragile on a computer. Because the rank jumps under a tiny change of one entry, rounding errors of size 10^{-16} can change it. Software (e.g. MATLAB's `rank`) therefore treats numbers below a small tolerance as zero, and the answer can depend on that tolerance.

Example (p. 97): the image of $T(x) = Ax$

Is the image of T a subspace? Yes, it is $\text{Col } A$

The image is $\{Ax : x \in \mathbb{R}^n\} = \{x_1 a_1 + \cdots + x_n a_n\}$, the set of all combinations of the columns: that is $\text{Col } A$, and every span is a subspace (9/15 notes, p. 85).

What is its dimension? $\text{rank}(A)$

$\dim(\text{image of } T) = \dim \text{Col } A = \# \text{pivots} = \text{rank } A$.

Rank Theorem (p. 98; Lay §2.9)

If $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear map, then

$$\dim(\text{image of } T) + \dim(\text{Ker } T) = n.$$

In matrix form, with $T(x) = Ax$: $\dim \text{Col } A + \dim \text{Nul } A = n$, i.e. $\text{rank } A + \dim \text{Nul } A = n$.

Proof (p. 98). Let $T(x) = Ax$, where A is the $m \times n$ standard matrix of T . Then

- image of $T = \text{Col } A$ (p. 97) and $\text{Ker } T = \text{Nul } A$ (9/15 notes, p. 84);
- $\dim \text{Col } A = \# \text{pivotal columns}$ and $\dim \text{Nul } A = \# \text{free variables} = \# \text{nonpivotal columns}$ (p. 95).

Any column is either pivotal or nonpivotal, and A has n columns, so the two numbers add up to n . $\square$

Example (p. 99)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}$, $T(x_1, x_2, x_3) = 2x_1 - x_2 + x_3$.

Is T linear? Yes: $T(x) = Ax$ with $A = [2 \ -1 \ 1]$

$Ax = [2 \ -1 \ 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2x_1 - x_2 + x_3 = T(x)$, and every matrix map is linear: $A(u+v) = Au + Av$, $A(cu) = cAu$.

Dimensions. A has one pivot, so $\dim(\text{image of } T) = \dim \text{Col } A = 1$ (the image is all of $\mathbb{R}$). By the Rank Theorem, $\dim \text{Ker } T = 3 - 1 = 2$: x_2, x_3 are free.

A basis of $\text{Ker } T$ (added): $x_1 = \frac{1}{2}x_2 - \frac{1}{2}x_3$, so

$$x = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}.$$

Check: $2 \cdot \frac{1}{2} - 1 + 0 = 0$ and $2 \cdot (-\frac{1}{2}) - 0 + 1 = 0$. $\checkmark$

5 k vectors in a k -dimensional subspace (pp. 99–102)

If you already know $\dim S = k$, then for k vectors in S it is enough to check *one* of “independent” or “spanning”.

Recall (p. 99). A basis of a subspace S is a set of independent vectors that span S :

$$\text{basis} = \text{independent} + \text{span}.$$

Theorem (p. 100; the Basis Theorem, Lay §2.9)

Let $v_1, \dots, v_k$ be vectors in a subspace S of $\mathbb{R}^n$. If $\dim S = k$, the following are equivalent:

- (1) $\{v_1, \dots, v_k\}$ is a basis of S ;
- (2) $S = \text{span}\{v_1, \dots, v_k\}$;
- (3) $\{v_1, \dots, v_k\}$ is independent.

Ex: any two nonproportional vectors in $\mathbb{R}^2$ form a basis of $\mathbb{R}^2$. Why

$\dim \mathbb{R}^2 = 2$. Two vectors are dependent exactly when one is a multiple of the other, so nonproportional vectors are independent: that is (3) with $k = 2$. The theorem gives (1).

Example: $(1, 1)$ and $(1, -1)$ from p. 93: we checked spanning by hand there, but the theorem makes that step unnecessary.

Proof (pp. 100–102). (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) hold by the definition of a basis, and (2) + (3) $\Rightarrow$ (1) also by definition. So it remains to show (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3). Let $A = [v_1 \ \dots \ v_k]$, an $n \times k$ matrix. Its pivotal columns form a basis of $\text{Col } A$ (p. 90), and $\text{Col } A = S$ by (2). But $\dim S = k$, so this basis has k vectors (p. 96): there are k pivots. So each of the k columns of A is pivotal, and the columns are independent.

(3) $\Rightarrow$ (2) (arguing by contradiction). Let $S_k := \text{span}\{v_1, \dots, v_k\} \subseteq S$. If S_k is smaller than S , then there is a v_{k+1} in S but not in S_k . As in the homework, $\{v_1, \dots, v_k, v_{k+1}\}$ is independent.

The homework fact: if $\{v_1, \dots, v_k\}$ is independent and $v_{k+1} \notin \text{span}\{v_1, \dots, v_k\}$, then $\{v_1, \dots, v_{k+1}\}$ is independent

The notes cite this as homework (“As in homework”, p. 101); proved here for reference.

Suppose $c_1 v_1 + \dots + c_k v_k + c_{k+1} v_{k+1} = 0$. If $c_{k+1} \neq 0$, divide by it:

$$v_{k+1} = -\frac{c_1}{c_{k+1}} v_1 - \dots - \frac{c_k}{c_{k+1}} v_k \in \text{span}\{v_1, \dots, v_k\},$$

which is false. So $c_{k+1} = 0$, and then $c_1 v_1 + \dots + c_k v_k = 0$ forces $c_1 = \dots = c_k = 0$ by independence.

Repeat the process for $S_{k+1} = \text{span}\{v_1, \dots, v_{k+1}\} \subseteq S$: if it is smaller than S , add a v_{k+2} , and so on. We get

$$S_k \subsetneq S_{k+1} \subsetneq S_{k+2} \subsetneq \dots \subseteq S \subseteq \mathbb{R}^n,$$

where each $S_i = \text{span}\{v_1, \dots, v_i\}$ ($i \geq k$) is spanned by *independent* vectors.

Why does this process terminate? $\mathbb{R}^n$ holds at most n independent vectors

At every step the list $v_1, \dots, v_i$ is independent, and by the Lemma below an independent list in $\mathbb{R}^n$ has at most n vectors. So we can add at most $n - k$ vectors; the process must stop, and it can only stop when $S_\ell = S$.

Lemma (p. 102)

$\mathbb{R}^n$ contains at most n independent vectors.

Proof. If $\{v_1, \dots, v_k\}$ in $\mathbb{R}^n$ is independent, then the $n \times k$ matrix $[v_1 \cdots v_k]$ has k pivots (every column is pivotal). Pivots lie in different rows, so $k \leq n$. $\square$

End of the proof (p. 102). So eventually $S_\ell = S$ for some $\ell > k$. Then $\{v_1, \dots, v_\ell\}$ is independent and spans S : it is a basis of S . So $\dim S = \ell > k = \dim S$, a contradiction. Hence $S_k = S$, which is (2). $\square$

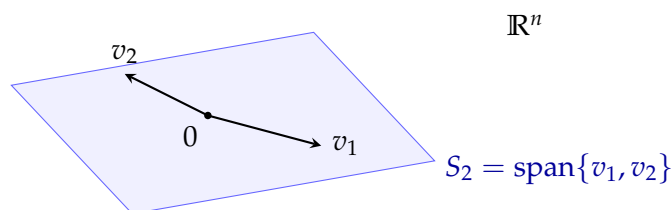


Figure 2: The sketch on p. 102: two independent vectors v_1, v_2 span a plane S_2 through 0 inside $\mathbb{R}^n$. Each step of the process adds a vector outside the current span, and the span grows by one dimension.

6 Every nonzero subspace has a basis (p. 103)

The same “keep adding a vector outside the span” process, started from one vector, builds a basis.

Theorem (p. 103)

Any nonzero subspace S of $\mathbb{R}^n$ has a basis.

Proof (p. 103). Pick any nonzero v_1 in S (possible since $S \neq \{0\}$). $\{v_1\}$ is independent because $v_1 \neq 0$.

- If $S = S_1 := \text{span}\{v_1\}$, then $\{v_1\}$ is a basis.
- If $S \neq S_1$, pick v_2 in S not in S_1 . By the homework fact (p. 101 above), $\{v_1, v_2\}$ is independent. If $S_2 := \text{span}\{v_1, v_2\}$ is S , then $\{v_1, v_2\}$ is a basis of S .
- If not, pick v_3 in S not in S_2 , etc.

The process terminates after at most n steps, because every list $v_1, \dots, v_i$ produced is independent and $\mathbb{R}^n$ contains at most n independent vectors (Lemma, p. 102). It stops only when $S_i = S$, and

then $\{v_1, \dots, v_i\}$ is a basis. □

Consequence. Together with p. 96, this shows that $\dim S$ is defined for every subspace S of $\mathbb{R}^n$ (with $\dim\{0\} = 0$), and $0 \leq \dim S \leq n$.

7 Vector spaces: a first look (p. 104)

Many other kinds of objects can be added and scaled like vectors in $\mathbb{R}^n$; this is the starting point for abstract vector spaces (Lay §4.1).

Recall (p. 104): vectors are things we can add and scale.

Example ($m \times n$ matrices). If A and B are $m \times n$ matrices, then $2A + B$ is again $m \times n$. So $m \times n$ matrices can be added and scaled, staying in the same set.

Example (polynomials). Let

$$f(x) = a_0 + a_1x + \cdots + a_nx^n, \quad g(x) = b_0 + b_1x + \cdots + b_mx^m, \quad \text{say } m < n.$$

Add coefficient by coefficient; the terms above degree m come from f alone:

$$\begin{aligned}(f + g)(x) &= (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_m + b_m)x^m \\ &\quad + a_{m+1}x^{m+1} + \cdots + a_nx^n, \\ 2f &= 2a_0 + 2a_1x + \cdots + 2a_nx^n.\end{aligned}$$

Worked example (added): $f = 1 + 2x + x^2$, $g = 3 - x$ ($n = 2$, $m = 1$): $f + g = 4 + x + x^2$ and $2f = 2 + 4x + 2x^2$.

Check at $x = 1$: $f(1) + g(1) = 4 + 2 = 6 = (4 + 1 + 1)$, and $2f(1) = 8 = 2 + 4 + 2$. ✓

Overlap with the 9/22 notes. Page 104 is also the first page of the 9/22 lecture, which continues with the formal definition of a vector space. It is covered fully here and again (with more detail) in the 9/22 notes, pp. 104–105.

Summary

Fact	Formula / recipe
Basis of Nul A	parametric vector form of $Ax = 0$; one vector per free variable
Basis of Col A	the pivotal columns of A
Coordinates	$x = c_1 b_1 + \cdots + c_k b_k$, $[x]_\beta = (c_1, \dots, c_k)$, unique
Dimension	number of vectors in any basis
$\dim \text{Col } A = \text{rank } A$	number of pivots
$\dim \text{Nul } A$	number of free variables (nonpivotal columns)
Rank Theorem	$\text{rank } A + \dim \text{Nul } A = n$ (number of columns)
Basis Theorem	if $\dim S = k$: k vectors in S independent $\iff$ spanning $\iff$ basis
Lemma	at most n independent vectors in $\mathbb{R}^n$

Corrections to the handwritten notes

Page	As written	Should be	Why
91	row 3 of A : $(2, 3, 1, 0)$	$(2, 3, -1, 0)$	otherwise row 3 reduces to $(0, 0, 2, 0)$, not zero, and $2a_1 - a_2 + a_3 \neq 0$
94	$\beta = \{b_1, \dots, b_n\},$ $(c_n - d_n)b_n$	$b_k, (c_k - d_k)b_k$	the basis has k vectors
95	$\dim S = \text{Nul } A$	$S = \text{Nul } A$ (so $\dim S = \dim \text{Nul } A$)	a dimension is a number, not a subspace
98	$T(x) = A$	$T(x) = Ax$	missing x
99	$[2 \ -1 \ -1]$	$[2 \ -1 \ 1]$	coefficient of x_3 in T is $+1$ (same free variables)