

Linear Algebra Notes — Lecture 10/22

Diagonalization: Proof, Examples, Powers, and Complex Eigenvalues

Lecture pages 222–236 · Lay §5.3, §5.5

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Overview

The 10/20 lecture ended by stating the diagonalization theorem: $A = PDP^{-1}$ with D diagonal exactly when there is a basis of eigenvectors (10/20 notes, p. 222). This lecture proves it and uses it:

1. the proof: $AP = PD$ says, column by column, $Av_i = \lambda_i v_i$;
2. examples, including a 3×3 matrix with a repeated eigenvalue that is still diagonalizable;
3. similarity is transitive, and similar matrices have the same characteristic polynomial (but not conversely);

4. powers: $A = PDP^{-1}$ gives $A^k = PD^kP^{-1}$;
5. diagonalizing over $\mathbb{C}$, and the real 2×2 case: a matrix with eigenvalues $a \pm ib$ is similar to $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, a rotation followed by a scaling.

Every blank and question in the notes is answered in a blue box. Typos are listed at the end.

Notation used throughout.

- $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ is a diagonal matrix; e_i is the i -th standard basis vector.
- $\bar{z}$ is the complex conjugate; $\bar{v}$ conjugates each entry (10/20 notes, p. 211).
- “Similar”: $A = PBP^{-1}$ with P invertible (10/20 notes, p. 221).

Recipe (from the proof on p. 223). To diagonalize an $n \times n$ matrix A :

1. find the eigenvalues (roots of $\det(A - \lambda I)$) and a basis of each eigenspace $\text{Nul}(A - \lambda I)$;
2. if these give n eigenvectors $v_1, \dots, v_n$ in total that form a basis, set $P = [v_1 \ \cdots \ v_n]$ and $D = \text{diag}(\lambda_1, \dots, \lambda_n)$, with λ_i the eigenvalue of v_i (same order!);
3. then $A = PDP^{-1}$. Otherwise A is not diagonalizable.

1 Recap: diagonalizable matrices (p. 222)

Page 222 is the last page of the 10/20 notes and is covered in full there (10/20 notes, p. 222, Sections 8–9). In brief:

- The proof that the matrices A_1, A_2 of one linear map $T : V \rightarrow V$ in two bases are similar: $A_1 = PA_2P^{-1}$ with $P = S_1S_2^{-1}$ (the change-of-basis matrix).

Definition (p. 222)

A matrix A is **diagonalizable** if $A = PDP^{-1}$ (P invertible) where $D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$ is a diagonal matrix.

Theorem (p. 222)

An $n \times n$ matrix A is diagonalizable over $\mathbb{R}$ (by definition: $A = PDP^{-1}$ with P, D real) $\iff \mathbb{R}^n$ has a basis of eigenvectors of A .

The final notes add a second form of the left side: $A = PDP^{-1}$ (P, D real) $\iff AP = PD$ with P invertible. (Multiply on the right by P for “ $\Rightarrow$ ”, by P^{-1} for “ $\Leftarrow$ ”.) The proof works with $AP = PD$.

2 Proof of the diagonalization theorem (p. 223)

$AP = PD$ is the same as the n equations $Av_i = \lambda_i v_i$, one per column; P is invertible exactly when its columns form a basis.

The tool used twice below is *column-by-column multiplication*: if $P = [p_1 \cdots p_n]$ then $AP = [Ap_1 \cdots Ap_n]$, and $Pe_i = p_i$ is column i of P (Lay §2.1).

Proof of “ $\Leftarrow$ ” (p. 223): $AP = [\lambda_1 v_1 \cdots \lambda_n v_n] = PD$

The notes: let $\{v_1, \dots, v_n\}$ be a basis of $\mathbb{R}^n$ with $Av_i = \lambda_i v_i$, $i = 1, \dots, n$, and let

$$P = [v_1 \cdots v_n], \quad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = [\lambda_1 e_1 \cdots \lambda_n e_n]$$

(column i of D has λ_i in position i and zeros elsewhere). Filling in the two blanks:

$$\begin{aligned} AP &= [Av_1 \cdots Av_n] = [\lambda_1 v_1 \cdots \lambda_n v_n], \\ PD &= [P\lambda_1 e_1 \cdots P\lambda_n e_n] = [\lambda_1 P e_1 \cdots \lambda_n P e_n] = [\lambda_1 v_1 \cdots \lambda_n v_n]. \end{aligned}$$

The first line uses $Av_i = \lambda_i v_i$; the second uses $Pe_i = v_i$. The notes join the two right-hand sides with “=”: $AP = PD$.

The columns of P form a basis of $\mathbb{R}^n$, so P is invertible (Invertible Matrix Theorem, Lay §2.3). Multiplying $AP = PD$ on the right by P^{-1} gives $A = PDP^{-1}$, with P and D real. ■

Proof of “ $\Rightarrow$ ” (p. 223), completed

Suppose $A = PDP^{-1}$ with P, D real, P invertible and D diagonal. Multiplying on the right by P gives $AP = PD$, and P is invertible, as the notes say. Let $P = [v_1 \cdots v_n]$ and $D = \text{diag}(\lambda_1, \dots, \lambda_n)$.

“As above”: the same two column computations give column i of $AP = Av_i$ and column i of $PD = \lambda_i v_i$. So $AP = PD$ says $Av_i = \lambda_i v_i$ for every i .

Why is $\{v_1, \dots, v_n\}$ a basis? Because P is invertible

The notes answer in three words: “ P is invertible.” In full: $P = [v_1 \cdots v_n]$ is an invertible $n \times n$ matrix, so by the Invertible Matrix Theorem (Lay §2.3) its columns are linearly independent and span $\mathbb{R}^n$. That is exactly: $\{v_1, \dots, v_n\}$ is a basis of $\mathbb{R}^n$.

In particular each $v_i \neq 0$ (a set containing 0 is dependent). Together with $Av_i = \lambda_i v_i$, each v_i is an eigenvector of A (with eigenvalue λ_i). So $\mathbb{R}^n$ has a basis of eigenvectors. ■

What the proof says. The columns of P are eigenvectors, and the diagonal entries of D are the matching eigenvalues, in the same order. So the diagonal of D lists all the eigenvalues of A , with multiplicity (similar matrices have the same characteristic polynomial: Section 5, p. 228).

3 Example: building A from P and D (p. 224)

Choosing P and D first produces a matrix whose eigenvectors and eigenvalues we know in advance.

Ex (p. 224)

The notes take $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. The inverse comes from the 2×2 formula $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ with $ad - bc = 2 - 1 = 1$. Then

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}.$$

(The notes write the top-right entry as 4, here and again further down the page; it is $(-2)(-1) + 2 \cdot 2 = 6$. See the corrections table.)

Eigenvalues $\lambda_1 = -1$, $\lambda_2 = 2$; eigenvectors $v_1 = (2, 1)$, $v_2 = (1, 1)$

The notes fill in $\lambda_1 = -1$, $\lambda_2 = 2$, $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. By the proof on p. 223, the eigenvalues are the diagonal entries of D , namely -1 and 2 , and the eigenvectors are the columns of P : $(2, 1)^T$ for -1 and $(1, 1)^T$ for 2 (each eigenspace is the span of that column).

Check: $A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -8+6 \\ -6+5 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} = -1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. ✓ Also trace $A = 1 = -1 + 2$ and $\det A = -20 + 18 = -2 = (-1) \cdot 2$ (10/20 notes, pp. 216–217). ✓

Can also write (p. 224). The order of the eigenvectors is a free choice, as long as D uses the same order. Swapping the columns of P and the diagonal entries of D :

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}.$$

Here $P^{-1} = \frac{1}{1-2} \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$.

Fill-in: the product is $\begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}$, the same A

$$\begin{bmatrix} 2 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2-2 & 4+2 \\ -2-1 & 4+1 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix} = A.$$

The notes write the product as $\begin{bmatrix} -4 & 4 \\ -3 & 5 \end{bmatrix}$, the same slip as above; the point, that it is the same A , stands. Meaning: a diagonalization is not unique. Any ordering of the eigenvectors works (and so does rescaling each one).

Ex (p. 224)

$A = PIP^{-1} = I$: any invertible P works.

This example is new in the final notes. $PIP^{-1} = PP^{-1} = I$, so the identity is diagonalizable with $D = I$ and *any* invertible P . In the language of the proof on p. 223: every nonzero vector is an

eigenvector of I (eigenvalue 1), so the columns of any invertible P are a basis of eigenvectors. Check ($P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$): $PIP^{-1} = PP^{-1} = I$. ✓

4 Repeated eigenvalues (pp. 225–227)

n distinct eigenvalues guarantee a basis of eigenvectors, but a basis can exist without them: what matters is whether each eigenspace is big enough.

Recall (p. 225). If an $n \times n$ matrix A has n distinct eigenvalues, then their eigenvectors form a basis (10/15 notes, p. 203). So such an A is diagonalizable.

Q (p. 225): if eigenvectors of A form a basis, must A have n distinct eigenvalues? No

The notes: “No. Ex $A = D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$, $P = I$, $A = IDI^{-1}$.” A diagonal matrix is diagonalizable with $P = I$: its eigenvectors $e_1, \dots, e_n$ ($De_i = d_i e_i$) form a basis. For a counterexample the d_i must not all be distinct, e.g. $D = \text{diag}(2, 2, 1)$: a basis of eigenvectors, but only two distinct eigenvalues.

The extreme case is the identity I_n ($n \geq 2$), all $d_i = 1$, the example on p. 224. Every nonzero vector is an eigenvector ($I_n v = 1 \cdot v$), so $e_1, \dots, e_n$ is a basis of eigenvectors. But I_n has only one eigenvalue, 1 (with multiplicity n). Indeed $I_n = I_n I_n I_n^{-1}$ is already diagonal.

The next example is a less trivial counterexample: a non-diagonal matrix with a repeated eigenvalue and a basis of eigenvectors.

Ex (pp. 225–227): $A = \begin{bmatrix} 2 & 0 & t \\ 0 & 2 & s \\ 0 & 0 & 1 \end{bmatrix}$, s, t real

Task: find all real t, s such that A is diagonalizable over $\mathbb{R}$, that is, $\mathbb{R}^3$ has a basis of eigenvectors of A .

Fill-in: $\lambda_1 = 2$, $\lambda_2 = 1$; multiplicities $m_1 = 2$, $m_2 = 1$

A is upper triangular, so $A - \lambda I$ is too, and its determinant is the product of the diagonal entries (10/15 notes, p. 195):

$$\det(A - \lambda I) = (2 - \lambda)(2 - \lambda)(1 - \lambda) = (1 - \lambda)^1(2 - \lambda)^2.$$

So, in the notes’ order, $\lambda_1 = 2$ with algebraic multiplicity $m_1 = 2$ and $\lambda_2 = 1$ with $m_2 = 1$ (10/20 notes, p. 213), for every s, t . Check: $m_1 + m_2 = 3 = n$. ✓

$\lambda_2 = 1$ (p. 226). $A - I = \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & s \\ 0 & 0 & 0 \end{bmatrix}$ is already in reduced echelon form.

Fill-in: $\text{Nul}(A - I)$ has dimension 1, spanned by $(-t, -s, 1)$

Pivots in columns 1 and 2; x_3 is free. So the dimension is $3 - 2 = 1$. The rows say

$$x_1 = -tx_3, \quad x_2 = -sx_3, \quad \text{so} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -t \\ -s \\ 1 \end{bmatrix}.$$

The eigenvector is $u = (-t, -s, 1)^T$.

Check: $Au = \begin{bmatrix} -2t+t \\ -2s+s \\ 1 \end{bmatrix} = \begin{bmatrix} -t \\ -s \\ 1 \end{bmatrix} = 1 \cdot u. \checkmark$

$\lambda_1 = 2$ (p. 226). $A - 2I = \begin{bmatrix} 0 & 0 & t \\ 0 & 0 & s \\ 0 & 0 & -1 \end{bmatrix}.$

Fill-in: $\text{Nul}(A - 2I)$ has dimension 2; eigenvectors $x_1e_1 + x_2e_2$, not both x_1, x_2 zero

The third row says $-x_3 = 0$ (as the notes write at the top of p. 227); then rows 1 and 2, $tx_3 = 0$ and $sx_3 = 0$, hold automatically. The matrix has rank 1 (one pivot: the notes mark the -1 in column 3), so the dimension is $3 - 1 = 2$, whatever s, t are. With x_1, x_2 free:

$$\text{Nul}(A - 2I) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

So the eigenvectors for 2 are the nonzero vectors $(x_1, x_2, 0)$. Check: $Ae_1 = 2e_1$ and $Ae_2 = 2e_2$ (the first two columns of A). $\checkmark$

Is there a basis of eigenvectors? Yes, for all real s, t

The notes: “Yes, $\begin{bmatrix} -t \\ -s \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is a basis.” Why these are independent: in the order e_1, e_2, u they are the columns of

$$Q = \begin{bmatrix} 1 & 0 & -t \\ 0 & 1 & -s \\ 0 & 0 & 1 \end{bmatrix},$$

which is upper triangular with 1's on the diagonal, so $\det Q = 1 \neq 0$ (equivalently, three pivots); reordering the vectors does not change independence. Three independent vectors in $\mathbb{R}^3$ form a basis. So A is diagonalizable over $\mathbb{R}$ for **all** real s, t . In the notes' order of the basis,

$$A = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} P^{-1}, \quad P = [u \ e_1 \ e_2] = \begin{bmatrix} -t & 1 & 0 \\ -s & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Check ($AP = PD$): $AP = [Au \ Ae_1 \ Ae_2] = [u \ 2e_1 \ 2e_2] = PD. \checkmark$

Why it works: the repeated eigenvalue 2 (multiplicity 2) has a 2-dimensional eigenspace, so it supplies two independent eigenvectors. Compare $\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$ (10/15 notes, p. 194): eigenvalue 3 of multiplicity 2, but only a 1-dimensional eigenspace, so no basis of eigenvectors. This

example is also an answer to the question on p. 225: a basis of eigenvectors, only two distinct eigenvalues.

5 Similarity (pp. 227–229)

Similarity is transitive, so diagonalizability passes to similar matrices; similar matrices share the characteristic polynomial, but sharing it does not make them similar.

Theorem (p. 227)

If A, B are similar and B, C are similar, then A, C are similar.

Proof (p. 227), completed: $A = (PQ)C(PQ)^{-1}$

The notes: $A = PBP^{-1}$ and $B = QCQ^{-1}$ with P, Q invertible; an arrow substitutes the second into the first, giving $PQCQ^{-1}P^{-1} = PQC(PQ)^{-1}$. In detail:

$$A = P(QCQ^{-1})P^{-1} = (PQ)C(Q^{-1}P^{-1}) = (PQ)C(PQ)^{-1},$$

using associativity and $(PQ)^{-1} = Q^{-1}P^{-1}$ (Lay §2.2). PQ is invertible (a product of invertible matrices), so A and C are similar. ■

Q (p. 228): if B is diagonalizable and $A = PBP^{-1}$, is A diagonalizable? Yes

The notes: $A = PQDQ^{-1}P^{-1} = PQD(PQ)^{-1}$, with D diagonal. In words: B diagonalizable means $B = QDQ^{-1}$ with D diagonal. So A is similar to B , and B is similar to D ; by the theorem on p. 227, A is similar to D :

$$A = (PQ)D(PQ)^{-1}.$$

So A is diagonalizable, with eigenvector matrix PQ . (Over $\mathbb{R}$: if P, Q, D are real, so is PQ .)

Example: $B = \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}$ from p. 224 is diagonalizable, so $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} B \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}$ is diagonalizable too.

Theorem (p. 228)

If A, B are similar, then $\det(A - \lambda I) = \det(B - \lambda I)$. In other words, A, B have the same eigenvalues (with the same multiplicities).

Proof (p. 228), completed

Let $A = PBP^{-1}$. Write $\lambda I = \lambda PP^{-1} = P(\lambda I)P^{-1}$. Then

$$\begin{aligned} \det(A - \lambda I) &= \det(PBP^{-1} - \lambda I) = \det(PBP^{-1} - P(\lambda I)P^{-1}) = \det(P(B - \lambda I)P^{-1}) \\ &= \det P \cdot \det(B - \lambda I) \cdot \det P^{-1} = \det(B - \lambda I). \end{aligned}$$

This is the notes' chain on p. 228. The third step factors out P on the left and P^{-1} on the right (distributive law). The fourth uses $\det(XY) = \det X \det Y$, and the last uses $\det P \cdot \det P^{-1} =$

$\det(PP^{-1}) = \det I = 1$ (10/8 notes, p. 174: $|PCP^{-1}| = |C|$); the notes cross out $\det P$ and $\det P^{-1}$ for this. ■

Exercise (p. 228): $\det(A - \lambda I) = \det(A^T - \lambda I)$

The exercise is new in the final notes. Transpose $A - \lambda I$: $(A - \lambda I)^T = A^T - \lambda I^T = A^T - \lambda I$, since transposing is linear and $I^T = I$. A matrix and its transpose have the same determinant (Lay §3.2, Theorem 5). So

$$\det(A^T - \lambda I) = \det((A - \lambda I)^T) = \det(A - \lambda I).$$

Meaning: A and A^T have the same eigenvalues, with the same multiplicities (but usually different eigenvectors).

Check ($A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$): $\det \begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}$ and $\det \begin{bmatrix} 1-\lambda & 3 \\ 2 & 4-\lambda \end{bmatrix}$ both equal $\lambda^2 - 5\lambda - 2$. ✓

Q (p. 229): if $\det(A - \lambda I) = \det(B - \lambda I)$, are A, B similar? No

The notes give two counterexamples, each written as a “ $\neq$ ”:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq P \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} P^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \neq P \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} P^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In each pair both matrices are triangular with the same diagonal, so they have the same characteristic polynomial: λ^2 for the first pair, $(1 - \lambda)^2$ for the second. But the only matrix similar to 0 is 0 ($P0P^{-1} = 0$), and the only matrix similar to I is I ($PIP^{-1} = PP^{-1} = I$). Since $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq 0$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \neq I$, neither pair is similar.

Ex (p. 229): a 2×2 matrix that is not diagonalizable: $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

The notes’ argument (via p. 228). Suppose $N = P \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} P^{-1}$. Similar matrices have the same eigenvalues, and N is triangular with diagonal 0, 0, so $d_1 = d_2 = 0$, as the notes write. Then $N = P \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} P^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which is false.

Second argument (eigenspace too small). $\text{Nul}(N - 0I) = \text{Nul } N = \text{span}\{e_1\}$ is 1-dimensional (N has one pivot). Every eigenvector is a multiple of e_1 , and two such vectors are never independent. So $\mathbb{R}^2$ has no basis of eigenvectors, and by the theorem on p. 222, N is not diagonalizable.

Check (a third way): $N^2 = 0$ but $N \neq 0$. If $N = PDP^{-1}$, then $PD^2P^{-1} = N^2 = 0$ forces $D^2 = 0$, so $D = 0$ and $N = 0$. ✓

Both arguments also work over $\mathbb{C}$, so N is not even diagonalizable over $\mathbb{C}$. The same reasoning shows that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (eigenvalues 1, 1, so D would be I) and $\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$ (10/15 notes, p. 194) are not diagonalizable.

6 Powers of a diagonalizable matrix (pp. 229–230)

If $A = PDP^{-1}$, then $A^k = PD^kP^{-1}$, and powers of a diagonal matrix are computed entry by entry.

Uses of diagonalizability. Ex (p. 229). Let $A = PDP^{-1}$ where $D = \text{diag}(\lambda_1, \dots, \lambda_n)$.

Fill-in: $D^k = \text{diag}(\lambda_1^k, \dots, \lambda_n^k)$ for $k > 0$

$$D^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}.$$

Reason: $De_i = \lambda_i e_i$, so applying D k times gives $D^k e_i = \lambda_i^k e_i$; that is column i of D^k . Check ($k = 2, n = 2$): $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}$. ✓

Fill-in: $A^3 = PD^3P^{-1}$ and $A^k = PD^kP^{-1}$

The inner $P^{-1}P$ pairs cancel:

$$A^3 = PDP^{-1}PDP^{-1}PDP^{-1} = PD(P^{-1}P)D(P^{-1}P)DP^{-1} = PDDDP^{-1} = PD^3P^{-1}.$$

In the same way, A^k is k copies of PDP^{-1} , with $k - 1$ cancelling pairs $P^{-1}P = I$:

$$A^k = PD^kP^{-1} = P \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix} P^{-1}.$$

Check with $A = \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}$ (p. 224), $k = 2$: directly $A^2 = \begin{bmatrix} 16-18 & -24+30 \\ 12-15 & -18+25 \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ -3 & 7 \end{bmatrix}$; by the formula, $PD^2P^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ -3 & 7 \end{bmatrix}$. ✓

Why it matters: computing A^k directly takes $k - 1$ matrix products; with the formula, it takes n powers of numbers and two products.

7 Diagonalizable over $\mathbb{C}$ (pp. 230–232)

Allowing complex P and D , matrices like rotations, with no real eigenvectors, become diagonalizable.

Definition (p. 230)

A is **diagonalizable over $\mathbb{C}$** if $A = PDP^{-1}$ for some complex matrices P, D (P invertible, D diagonal).

The proof on p. 223 works word for word with $\mathbb{C}^n$ in place of $\mathbb{R}^n$: A is diagonalizable over $\mathbb{C} \iff \mathbb{C}^n$ has a basis of (complex) eigenvectors of A .

Ex (pp. 230–231): $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, a, b real, $b \neq 0$

$$\det(A - \lambda I) = \det \begin{bmatrix} a - \lambda & -b \\ b & a - \lambda \end{bmatrix} = (a - \lambda)^2 + b^2.$$

The notes rewrite this as $(\lambda - a)^2 + b^2 = 0$, i.e. $(\lambda - a)^2 = -b^2$, so $\lambda - a = \pm ib$. So $\lambda_{\pm} = a \pm ib$. They are distinct eigenvalues “as $b \neq 0$ ”, in the notes’ words, and not real. (Conjugate pair, as on p. 212 of the 10/20 notes.)

Fill-in: $v_- = (1, i)$ for $\lambda_- = a - ib$

$A - \lambda_- I = A - (a - ib)I = \begin{bmatrix} ib & -b \\ b & ib \end{bmatrix}$, as in the notes. Row 1 says $ibz_1 - bz_2 = 0$; divide by $b \neq 0$: $z_2 = iz_1$. Row 2, (b, ib) , is $-i$ times row 1, $(ib, -b)$, so it says the same. With $z_1 = 1$:

$$v_- = \begin{bmatrix} 1 \\ i \end{bmatrix}. \quad \text{Check: } Av_- = \begin{bmatrix} a - ib \\ b + ia \end{bmatrix} = (a - ib) \begin{bmatrix} 1 \\ i \end{bmatrix}, \text{ since } (a - ib)i = ia + b. \checkmark$$

Fill-in: $v_+ = (1, -i) = \overline{v_-}$ for $\lambda_+ = a + ib$

By the conjugate-pair theorem (10/20 notes, p. 212), A is real, so $\overline{v_-} = (1, -i)$ is an eigenvector for $\overline{\lambda_-} = a + ib$. Directly: $A - \lambda_+ I = \begin{bmatrix} -ib & -b \\ b & -ib \end{bmatrix}$, row 1 gives $z_2 = -iz_1$.

Check: $Av_+ = \begin{bmatrix} a + ib \\ b - ia \end{bmatrix} = (a + ib) \begin{bmatrix} 1 \\ -i \end{bmatrix}$, since $(a + ib)(-i) = b - ia$. $\checkmark$

Is $\{v_+, v_-\}$ a basis of $\mathbb{C}^2$? Yes

The notes: “Yes, since they are independent.” Why they are independent: they are eigenvectors for the distinct eigenvalues $a + ib \neq a - ib$, so they are independent (10/15 notes, p. 203; the proof is the same over $\mathbb{C}$), and two independent vectors in the 2-dimensional space $\mathbb{C}^2$ form a basis. Directly: neither is a multiple of the other, since $v_+ = cv_-$ would need $c = 1$ (first entries) and $-i = i$ (second entries). Or: $\det[v_+ \ v_-] = \det \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} = i + i = 2i \neq 0$.

Conclusion (p. 231). So if $P = [v_+ \ v_-]$ and $D = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$, then $AP = PD$, and so $A = PDP^{-1}$: A is diagonalizable over $\mathbb{C}$. It is *not* diagonalizable over $\mathbb{R}$: its eigenvalues are not real, so by the theorem on p. 211 of the 10/20 notes it has no real eigenvectors at all.

Ex (p. 232): $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

This is the 90° clockwise rotation from the 10/20 notes (pp. 209–210), and the previous example with $a = 0$, $b = -1$. The notes label the eigenvalues $\lambda_{\pm} = \pm i$ and use $D = \text{diag}(i, -i)$. The eigenvectors: $A(1, i)^T = (i, -1)^T = i(1, i)^T$ and $A(1, -i)^T = (-i, -1)^T = -i(1, -i)^T$. So $v_+ = (1, i)$ goes with $\lambda_+ = i$, and $v_- = (1, -i)$ with $\lambda_- = -i$.

Fill-in: $P = [v_+ \ v_-] = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}, P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$

The notes write $P = [v_+ \ v_-] = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$, copying $v_+ = (1, -i)$ and $v_- = (1, i)$ from p. 231. But those formulas go with $\lambda_+ = a + ib$, which is $-i$ here (since $b = -1$), while the notes now call i the eigenvalue λ_+ . With $D = \text{diag}(i, -i)$ the columns must be swapped: the notes' P gives $PDP^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = A^T$, not A . (Their P is right with $D = \text{diag}(-i, i)$. See the corrections table.) The answers for $k = 4$ and $k = 2026$ are unaffected, since D^k is a multiple of I for even k .

The columns are $v_+ = (1, i)$ and $v_- = (1, -i)$. By the 2×2 inverse formula, with $\det P = -i - i = -2i$,

$$P^{-1} = \frac{1}{-2i} \begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \quad \left(\text{using } \frac{1}{-2i} = \frac{i}{2} \right).$$

Check: $PP^{-1} = \frac{1}{2} \begin{bmatrix} 1+i & -i+i \\ i-i & 1+1 \end{bmatrix} = I$, and $AP = \begin{bmatrix} i & -i \\ -1 & -1 \end{bmatrix} = PD$ with $D = \text{diag}(i, -i)$. ✓

Fill-in: $A^k = P \begin{bmatrix} i^k & 0 \\ 0 & (-i)^k \end{bmatrix} P^{-1} = \begin{bmatrix} \frac{i^k + (-i)^k}{2} & \frac{-i(i^k - (-i)^k)}{2} \\ \frac{i(i^k - (-i)^k)}{2} & \frac{i^k + (-i)^k}{2} \end{bmatrix}$

By Section 6, $A^k = PD^kP^{-1}$ with $D^k = \text{diag}(i^k, (-i)^k)$. Multiplying out with $x = i^k, y = (-i)^k$:

$$P \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} P^{-1} = \begin{bmatrix} x & y \\ ix & -iy \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} = \frac{1}{2} \begin{bmatrix} x+y & -i(x-y) \\ i(x-y) & x+y \end{bmatrix}.$$

Since i^k cycles through $i, -1, -i, 1$, this gives $A^k = A, -I, -A, I$ for $k \equiv 1, 2, 3, 0 \pmod{4}$. Meaning: four 90° rotations are the identity.

Check ($k = 1$): $x + y = 0, x - y = 2i$, so the formula gives $\frac{1}{2} \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = A$. ✓

Fill-in: $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}^4 = I$

The notes: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Entry by entry: $i^4 = (i^2)^2 = (-1)^2 = 1$ and $(-i)^4 = (-1)^4 i^4 = 1$. So $D^4 = I$ and $A^4 = PIP^{-1} = I$. *Check:* $A^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$, so $A^4 = (-I)^2 = I$. ✓

Fill-in: $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}^{2026} = -I$

The notes reduce the exponent: $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}^{2026} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, because the fourth power is I and $2026 = 4 \cdot 506 + 2$. Entry by entry, $i^{2026} = (i^4)^{506} i^2 = -1$, and $(-i)^{2026} = (-1)^{2026} i^{2026} = -1$. So $D^{2026} = -I$ and

$$A^{2026} = P(-I)P^{-1} = -I.$$

Check: $A^{2026} = (A^4)^{506} A^2 = I \cdot (-I) = -I$. ✓

“Generically, an $n \times n$ real matrix is diagonalizable over $\mathbb{C}$ but not over $\mathbb{R}$ ” (p. 232). “Generically” means: for a matrix with randomly chosen entries, with probability 1. Such a matrix has n distinct (complex) eigenvalues, so it is diagonalizable over $\mathbb{C}$ (10/15 notes, p. 203, used over $\mathbb{C}$). My reading of “but not over $\mathbb{R}$ ” is “but not necessarily over $\mathbb{R}$ ”: whether it is diagonalizable over $\mathbb{R}$ depends on whether all its eigenvalues are real. Both cases occur for a set of matrices with positive probability: the rotation-scaling matrices $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $b \neq 0$, and any small change of them have non-real eigenvalues, while $\text{diag}(1, 2)$ and any small change of it have two distinct real eigenvalues. (Addition; this is not in the notes.)

8 Real matrices with complex eigenvalues (pp. 233–236)

A real 2×2 matrix with eigenvalues $a \pm ib$ is, in a suitable real basis, a rotation followed by a scaling by $r = \sqrt{a^2 + b^2}$.

Fact (p. 233)

If A is an $n \times n$ real matrix with n real eigenvalues (possibly not distinct), then

$$A \text{ is diagonalizable over } \mathbb{C} \iff A \text{ is diagonalizable over } \mathbb{R}.$$

The notes: “Proof probably will omit. This is not that hard.”

Proof sketch (addition, not in the notes). “ $\Leftarrow$ ” is clear: real P, D are also complex. For “ $\Rightarrow$ ”: $\mathbb{C}^n$ has a basis of eigenvectors, each belonging to a real eigenvalue λ . For real λ , $A - \lambda I$ is a real matrix, so its row reduction uses only real numbers, and $\text{Nul}(A - \lambda I)$ has a real basis, of the same size d_λ over $\mathbb{R}$ as over $\mathbb{C}$. At most d_λ of the complex basis vectors lie in that eigenspace (they are independent), so $\sum_\lambda d_\lambda \geq n$. Put the real bases of all eigenspaces together: the result is independent, because eigenvectors for distinct eigenvalues are independent (Lay §5.3, Theorem 7, extending 10/15 notes, p. 203). So we get at least n , hence exactly n , independent real eigenvectors in $\mathbb{R}^n$: a real basis of eigenvectors. ■

Theorem (p. 233)

Let A be a 2×2 real matrix with a non-real eigenvalue $a - ib$, $b \neq 0$. Then A is similar to

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Notation warning. In this proof the notes write $v_- = R + iI$ where R and I are *real vectors* (the real and imaginary parts of v_- , as with $v = a + ib$ in the 10/20 notes, p. 211). Here I is *not* the identity matrix. For example, $v_- = (1, i) = (1, 0) + i(0, 1)$ has $R = (1, 0)$, $I = (0, 1)$.

Proof (pp. 233–235). Let v_- be a complex eigenvector of A for $a - ib$, and write $v_- = R + iI$ with $R, I \in \mathbb{R}^2$. Then

$$AR + iAI = A(R + iI) = (a - ib)(R + iI) = (aR + bI) + i(aI - bR),$$

multiplying out with $i \cdot (-ib) = b$. A is real, so AR , AI , $aR + bI$, $aI - bR$ are all real vectors.

Comparing real and imaginary parts:

$$AR = aR + bI = [R \ I] \begin{bmatrix} a \\ b \end{bmatrix}, \quad AI = -bR + aI = [R \ I] \begin{bmatrix} -b \\ a \end{bmatrix}$$

(a matrix times a vector is the combination of its columns with those weights). Putting the two columns side by side,

$$A[R \ I] = [R \ I] \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad P = [R \ I].$$

Is $P = [R \ I]$ invertible? Yes

For a 2×2 matrix, the columns are dependent exactly when one is a multiple of the other. The notes test this: "If say $R = cI$ for real c , then $v_- = R + iI = (c + i)I$. So I is a real eigenvector for A and $a - ib$." Indeed $c + i \neq 0$, so $I = \frac{1}{c+i}v_-$ is a nonzero multiple of an eigenvector, hence an eigenvector (with the same eigenvalue $a - ib$).

The other case is $I = cR$ for real c (this includes $I = 0$); then $v_- = (1 + ic)R$, so R is a real eigenvector in the same way.

Can this be? No

The notes: "No." The reason: a real matrix with a non-real eigenvalue has no real eigenvectors for that eigenvalue (Theorem, 10/20 notes, p. 211): if v is real, Av is real but $(a - ib)v$ is not. Here $b \neq 0$, so $a - ib$ is not real. So neither column of P is a multiple of the other, the columns are independent, and P is invertible (Invertible Matrix Theorem). ■

Thus (p. 235). Multiplying $AP = P \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ on the right by P^{-1} :

$$A = [R \ I] \begin{bmatrix} a & -b \\ b & a \end{bmatrix} [R \ I]^{-1}.$$

Geometric meaning of $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ as a transformation of $\mathbb{R}^2$ (p. 235)

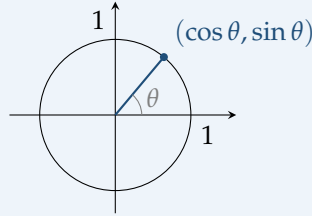
Let $r = \sqrt{a^2 + b^2} = |a - ib|$, the absolute value of the eigenvalue, as the notes add ($r > 0$, since $b \neq 0$). Factor it out:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r \begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix}.$$

Fill-in: $\left(\frac{a}{r}\right)^2 + \left(\frac{b}{r}\right)^2 = 1$

$$\left(\frac{a}{r}\right)^2 + \left(\frac{b}{r}\right)^2 = \frac{a^2 + b^2}{r^2} = \frac{a^2 + b^2}{a^2 + b^2} = 1.$$

So $(a/r, b/r)$ is a point on the unit circle, and there is an angle θ with $a/r = \cos \theta$ and $b/r = \sin \theta$, as the notes say. Their sketch:



The figure shows the unit circle and the point $(\cos \theta, \sin \theta) = (a/r, b/r)$ on it, at angle θ from the positive x_1 -axis.

Hence

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

The matrix on the right is the rotation by θ counterclockwise (Lay §1.9): its first column, the image of e_1 , is $(\cos \theta, \sin \theta)$, i.e. e_1 turned by θ ; its second column is e_2 turned by θ . Multiplying by r scales every vector by r . So, as the notes put it,

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \text{rotate by } \theta = \arccos(a/r), \text{ then scale by } r.$$

(Fine print: $\arccos$ gives an angle in $[0, \pi]$, which is the right θ when $b > 0$. When $b < 0$, the rotation is clockwise: $\theta = -\arccos(a/r)$. For example $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, with $a = 0$, $b = -1$, is the 90° clockwise rotation.) Rotating and scaling commute, so “then” could also be “first”.

Meaning (p. 236). This gives a geometric meaning of complex eigenvalues and eigenvectors of a real 2×2 matrix: with $P = [R \ I]$ built from the eigenvector $v_- = R + iI$,

$$A = P \begin{bmatrix} \text{rotate} \\ \text{then} \\ \text{scale} \end{bmatrix} P^{-1}.$$

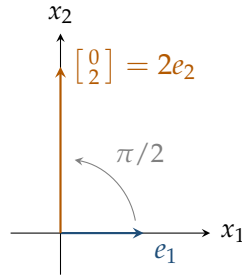
In the coordinates of the (real) basis $\{R, I\}$, A rotates by the angle θ with $\cos \theta = a/r$, $\sin \theta = b/r$, and scales by $r = \sqrt{a^2 + b^2} = |a \pm ib|$. The eigenvalue gives the angle and the scale factor; the eigenvector gives the basis.

Ex (p. 236)

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 16 & -26 \\ 10 & -16 \end{bmatrix}.$$

Here $P = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ (so $R = (2, 1)$, $I = (3, 2)$), $P^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ (determinant $4 - 3 = 1$), and the middle matrix has $a = 0$, $b = 2$. (The notes write the top-right entry as 26; it is $6 \cdot (-3) + (-4) \cdot 2 = -26$. See the corrections table.)

The notes' arrow marks the middle matrix as the “rotate then scale” factor of $A = P[\dots]P^{-1}$. The middle matrix: $\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$: rotate by $\pi/2$ (here $r = 2$, $\cos \theta = 0$, $\sin \theta = 1$), scale by 2. The notes' sketch shows this:



The picture shows e_1 (blue) turned by 90° counterclockwise and stretched to length 2 (orange).

In the original coordinates the notes compute $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix}$, the first column of A . This is not a rotation of e_1 (its length is $\sqrt{356}$, not 2): A rotates and scales only in the coordinates of the basis $\{R, I\}$.

Check (eigenvalues and eigenvector). By p. 228, A has the eigenvalues of $\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$, namely $\pm 2i$. Indeed $\text{trace } A = 0 = 2i + (-2i)$ and $\det A = -256 + 260 = 4 = (2i)(-2i)$. ✓ And $v_- = R + iI = (2 + 3i, 1 + 2i)$ should be an eigenvector for $a - ib = -2i$:

$$Av_- = \begin{bmatrix} 16(2 + 3i) - 26(1 + 2i) \\ 10(2 + 3i) - 16(1 + 2i) \end{bmatrix} = \begin{bmatrix} 6 - 4i \\ 4 - 2i \end{bmatrix} = -2i \begin{bmatrix} 2 + 3i \\ 1 + 2i \end{bmatrix}. \checkmark$$

Summary

Fact	Formula
Diagonalization	$A = PDP^{-1} \iff AP = PD \iff Av_i = \lambda_i v_i$, $P = [v_1 \cdots v_n]$ invertible
Distinct eigenvalues	n distinct $\Rightarrow$ diagonalizable (converse false: I_n)
Repeated eigenvalue	need $\dim \text{Nul}(A - \lambda I)$ big enough (p. 226 example: yes)
Similarity	transitive; keeps diagonalizability and the char. polynomial
Same char. polynomial	does <i>not</i> imply similar: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ vs. 0
Powers	$A^k = PD^k P^{-1}$, $D^k = \text{diag}(\lambda_i^k)$
$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $b \neq 0$	$\lambda = a \pm ib$; diagonalizable over $\mathbb{C}$, not over $\mathbb{R}$
Real 2×2 , eigenvalue $a - ib$	$A = [R \ I] \begin{bmatrix} a & -b \\ b & a \end{bmatrix} [R \ I]^{-1}$, $v_- = R + iI$
Geometry	$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ = rotate by θ , scale by $r = \sqrt{a^2 + b^2}$

Corrections to the handwritten notes

Page	As written	Should be	Why
224	$A = \begin{bmatrix} -4 & 4 \\ -3 & 5 \end{bmatrix}$ (twice)	$\begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}$	$(-2)(-1) + 2 \cdot 2 = 6$
226	$x_1 = 1$	$\lambda_2 = 1$	the eigenvalue 1 is λ_2 in the notes' labels
226	$\lambda_2 = 2$	$\lambda_1 = 2$	the notes set $\lambda_1 = 2$ at the top of p. 226
232	$P = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$ with $D = \text{diag}(i, -i)$	$P = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$	$(1, -i)$ is the eigenvector for $-i$; the notes' $PDP^{-1} = A^T$
236	top-right entry 26	-26	$6(-3) + (-4)2 = -26$