Linear Algebra Notes — Lecture 10/29

Dot Product, Length, Orthogonality and Orthogonal Complements

Lecture pages 257–272 · Lay §6.1–6.2

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Overview

This lecture starts a new chapter. Until now $\mathbb{R}^n$ had no notion of length or angle. The *dot product* $u \cdot v = u^T v$ supplies both, in a way that is (a) easy to compute and (b) agrees with the familiar distance and angle in $\mathbb{R}^2$ and $\mathbb{R}^3$. The lecture covers:

1. the dot product, length (norm), distance, unit vectors, and the angle between two vectors;
2. orthogonality: $u \cdot v = 0$, the Pythagorean theorem, and orthonormal sets;
3. the orthogonal complement $W^\perp$, with $\dim W + \dim W^\perp = n$;
4. $(\text{Row } A)^\perp = \text{Nul } A$ and $(\text{Col } A)^\perp = \text{Nul}(A^T)$: the four fundamental subspaces;
5. orthonormal sets are independent, and coordinates in an orthonormal basis are just dot products.

The goals for the coming lectures (p. 257) are: ① linear maps $T : \mathbb{R}^n \to \mathbb{R}^n$ that preserve distances and angles, ② orthogonal projections, ③ best fit (least squares), ④ inner product spaces, ⑤ complex dot and inner products.

Every blank and question in the notes is answered in a blue box. Typos are listed at the end.

Notation used throughout.

- Vectors in $\mathbb{R}^n$ are columns; u^T is the row with the same entries.
- $\|u\|$ is the length (2-norm) of u; $u \perp v$ means $u \cdot v = 0$.
- Row A is the row space of A; $W^\perp$ is the orthogonal complement of a subspace W.

1 Dot product and length (pp. 257–258)

The dot product multiplies matching entries and adds; the length of u is $\sqrt{u \cdot u}$, which is the usual length by the Pythagorean theorem.

Definition (dot product, p. 257)

For u, v in $\mathbb{R}^n$,

$$u \cdot v = u^T v = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + \cdots + u_n v_n.$$

($u^T v$ is a $1 \times n$ matrix times an $n \times 1$ matrix, so a 1×1 matrix, i.e. a number.)

Definition (length, p. 257)

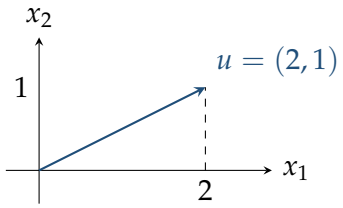
The **length** of u (or **2-norm** of u) is $\|u\| = \sqrt{u \cdot u} = \sqrt{u_1^2 + \cdots + u_n^2}$.

The square root makes sense because $u \cdot u$ is a sum of squares, so $u \cdot u \geq 0$.

Definition (orthogonal, p. 258)

u, v in $\mathbb{R}^n$ are **orthogonal** if $u \cdot v = 0$.

Ex (p. 258): $u = (2,1)$ in $\mathbb{R}^2$

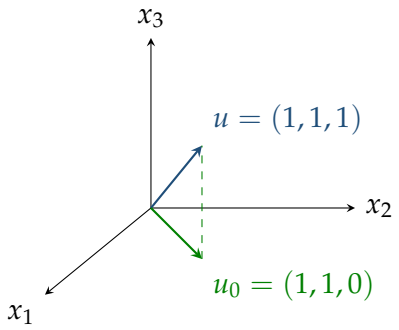


Fill-in: $\|u\|^2 = 5$

The notes: $\|u\|^2 = \begin{bmatrix}2\\1\end{bmatrix} \cdot \begin{bmatrix}2\\1\end{bmatrix} = 2\cdot 2 + 1\cdot 1 = 5$, so $\|u\| = \sqrt{5}$. **Why it makes sense:** u is the hypotenuse of a right triangle with legs 2 (along x_1) and 1 (up), so the Pythagorean theorem gives $2^2 + 1^2$ too. ✓

Ex (p. 258): $u = (1,1,1)$ in $\mathbb{R}^3$

The notes draw u and its "shadow" $u_0 = (1,1,0)$ in the x_1x_2-plane (in green). The segment from u_0 up to u is vertical, of length 1.



Fill-in: $\|u\|^2 = \|u_0\|^2 + 1^2 = 2 + 1 = 3$

u_0, the vertical segment $(0,0,1)$ and u form a right triangle (the segment is perpendicular to the x_1x_2-plane). So

$$\|u\|^2 = \|u_0\|^2 + 1^2 = 1^2 + 1^2 + 1^2 = 3, \qquad \|u\| = \sqrt{3}$$

(as in the notes). *Check:* directly, $u \cdot u = 1 + 1 + 1 = 3$. ✓ So in $\mathbb{R}^3$ the formula $\|u\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$ is the Pythagorean theorem used twice: this is goal (b) on p. 257.

Ex (p. 258): $e_i \cdot e_j$, where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$

Fill-in: $e_i \cdot e_j = 0$ if $i \neq j$, and 1 if $i = j$

e_i has a 1 in position i and 0 elsewhere. If $i \neq j$, no position has nonzero entries in both, so every product $(e_i)_k (e_j)_k$ is 0. If $i = j$, the only nonzero product is $1 \cdot 1$. So distinct standard basis vectors are orthogonal and each has length 1: $\{e_1, \ldots, e_n\}$ is *orthonormal* (definition on p. 259).

Ex (p. 258): $0 \cdot v$

Fill-in: $0 \cdot v = 0$

Every term is $0 \cdot v_k = 0$. So the zero vector is orthogonal to every vector.

2 Properties of the dot product (p. 259)

The dot product is symmetric, linear in each slot, and positive: these three rules are all later proofs use.

Properties of the dot product (p. 259)

① (symmetry) $u \cdot v = v \cdot u$;

② (linearity) $(c_1 u_1 + c_2 u_2) \cdot u = c_1 (u_1 \cdot u) + c_2 (u_2 \cdot u)$;

③ (positivity) $u \cdot u \geq 0$, and $u \cdot u = 0 \iff u = 0$.

Why they hold. ①: $u_k v_k = v_k u_k$ in each term. ②: $(c_1 u_1 + c_2 u_2) \cdot u = (c_1 u_1 + c_2 u_2)^T u = c_1 u_1^T u + c_2 u_2^T u$, by the distributive law for matrix products.

Fill-in: $u \cdot (c_1 u_1 + c_2 u_2) = c_1 (u \cdot u_1) + c_2 (u \cdot u_2)$

Use symmetry to move the combination to the left, apply linearity, then symmetry again:

$$u \cdot (c_1 u_1 + c_2 u_2) \overset{(1)}{=} (c_1 u_1 + c_2 u_2) \cdot u \overset{(2)}{=} c_1 (u_1 \cdot u) + c_2 (u_2 \cdot u) \overset{(1)}{=} c_1 (u \cdot u_1) + c_2 (u \cdot u_2).$$

So the dot product is linear in the second slot as well.

Fill-in: $u \cdot u = 0 \iff u = 0$

$u \cdot u = u_1^2 + \cdots + u_n^2$ is a sum of squares, so it is ≥ 0, and it is 0 only if every $u_k^2 = 0$, i.e. every $u_k = 0$. Conversely $0 \cdot 0 = 0$.

Definition (orthonormal, p. 259)

Vectors $u_1, \ldots, u_k$ in $\mathbb{R}^n$ are **orthonormal** if $u_i \cdot u_j = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$

In words: they are pairwise orthogonal, and each has length 1 ($\|u_i\|^2 = u_i \cdot u_i = 1$). Example: $e_1, \ldots, e_n$ (p. 258).

3 Using $\|u - v\|^2$ to find a dot product (p. 260)

Expanding $\|u - v\|^2$ with the dot product rules connects lengths to dot products.

Ex (p. 260): u, v in $\mathbb{R}^n$ with $\|u\| = 5$, $\|v\| = 2$, $\|u - v\| = 3$. Find $u \cdot v$

Answer: $u \cdot v = 10$

Expand using linearity and symmetry (p. 259):

$$\|u - v\|^2 = (u - v) \cdot (u - v) = u \cdot u - u \cdot v - v \cdot u + v \cdot v = \|u\|^2 - 2\, u \cdot v + \|v\|^2.$$

So $9 = 25 - 2\, u \cdot v + 4$, i.e. $2\, u \cdot v = 20$, and $u \cdot v = 10$.
Check: such vectors exist: $v = (2, 0)$ and $u = (5, 0)$ give $\|u\| = 5$, $\|v\| = 2$, $\|u - v\| = 3$ and $u \cdot v = 10$. ✓
Meaning: here $3 = 5 - 2$, so the "triangle" in the notes' picture is flat: u and v point the same way. Indeed $u \cdot v = 10 = \|u\|\|v\|$, the largest value possible (this is the Cauchy–Schwarz inequality, Lay §6.7, not yet in the course).

4 Unit vectors and distance (p. 261)

Scaling a vector by c scales its length by $|c|$, so dividing a nonzero vector by its length gives a unit vector.

Definition (unit vector, p. 261)

u is a **unit vector** if $\|u\| = 1$.

Fill-in (scaling): $\|cu\| = |c|\,\|u\|$ for u in $\mathbb{R}^n$, c in $\mathbb{R}$

$\|cu\|^2 = (cu) \cdot (cu) = c^2(u \cdot u) = c^2\|u\|^2$ by linearity in each slot. Take square roots: $\|cu\| = \sqrt{c^2}\,\|u\| = |c|\,\|u\|$. (Note $|c|$, not c: $\|-u\| = \|u\|$.)

Note (p. 261): for $u \neq 0$, $\frac{u}{\|u\|}$ has length 1

Use scaling with $c = 1/\|u\| > 0$ (this needs $u \neq 0$, so that $\|u\| \neq 0$ by positivity): $\left\| \frac{1}{\|u\|} u \right\| = \frac{1}{\|u\|} \|u\| = 1$.
This is called **normalizing** u. *Example:* $u = (2,1)$ from p. 258 gives the unit vector $(2,1)/\sqrt{5}$; check: $(4+1)/5 = 1$. ✓

Definition (distance, p. 261)

The **distance** between u, v in $\mathbb{R}^n$ is $\|u - v\|$.

In $\mathbb{R}^2$ this is the usual formula $\sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$.

5 Pythagoras and non-degeneracy (p. 262)

Orthogonal vectors are exactly those for which the Pythagorean theorem holds; and only the zero vector is orthogonal to everything.

Theorem (Pythagoras, p. 262)

$u \cdot v = 0 \iff \|u \pm v\|^2 = \|u\|^2 + \|v\|^2$.

Proof: expand $\|u \pm v\|^2 = (u \pm v) \cdot (u \pm v)$

The notes: $(u \pm v) \cdot (u \pm v) = u \cdot u \pm 2\, u \cdot v + v \cdot v$, and cross out $\|u\|^2$ and $\|v\|^2$ on both sides. In words: as on p. 260, $\|u \pm v\|^2 = (u \pm v) \cdot (u \pm v) = \|u\|^2 \pm 2\, u \cdot v + \|v\|^2$. So $\|u \pm v\|^2 = \|u\|^2 + \|v\|^2$ exactly when $2\, u \cdot v = 0$, i.e. $u \cdot v = 0$. (This works for either sign.) ■
Check: $u = (2,0)$, $v = (0,1)$: $u \cdot v = 0$ and $\|u + v\|^2 = 4 + 1 = 5 = \|u\|^2 + \|v\|^2$. ✓

Theorem (non-degeneracy of the dot product, p. 262)

If $u \cdot v = 0$ for every v in $\mathbb{R}^n$, then $u = 0$.

Fill-in: $u = 0$. Proof: take $v = u$

The notes: "Take $v = u$: $0 = u \cdot u \Rightarrow u = 0$." The hypothesis holds for every v, in particular for $v = u$: $u \cdot u = 0$. By positivity (p. 259), $u = 0$. ■
(Alternatively, take $v = e_i$: $u \cdot e_i = u_i = 0$ for each i.)

6 Angles and perpendicular vectors (p. 263)

The angle between nonzero u, v is defined by $\cos\theta = \dfrac{u \cdot v}{\|u\|\,\|v\|}$; dot product zero means perpendicular.

Definition (angle, p. 263)

The **angle made (or formed) by nonzero** u, v in $\mathbb{R}^n$ is the θ with $0 \le \theta \le \pi$ such that

$$\cos\theta = \frac{u \cdot v}{\|u\|\,\|v\|}.$$

Note (p. 263). $\theta \mapsto \cos\theta$ is one-to-one for $0 \le \theta \le \pi$ (it decreases from 1 to -1, as the notes' graph shows), so $\cos\theta$ determines θ.

Fine print. For such a θ to exist, the fraction must lie between -1 and 1, i.e. $|u \cdot v| \le \|u\|\,\|v\|$. That is the Cauchy–Schwarz inequality (Lay §6.7), which the notes don't prove yet. **Why it matches $\mathbb{R}^2$ (bottom of p. 262):** for the triangle with sides $a = \|u\|$, $b = \|v\|$ and $c = \|v - u\|$, the law of cosines $c^2 = a^2 + b^2 - 2ab\cos\theta$ gives $\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos\theta$, and the expansion on p. 260 gives the same with $u \cdot v$ in place of $\|u\|\|v\|\cos\theta$ (Lay §6.1).

Definition (perpendicular, p. 263)

Nonzero u, v are **perpendicular** if they make angle $\pi/2$.

Thus $u \cdot v = 0 \iff u = 0$ or $v = 0$ or u, v are perpendicular, and we write $u \perp v$. (For nonzero u, v: $\cos\theta = 0 \iff \theta = \pi/2$ on $[0, \pi]$.) So "orthogonal" (p. 258) is "perpendicular", except that it also allows the zero vector.

7 The vectors orthogonal to one vector (p. 264)

For $u \neq 0$, the vectors orthogonal to u form a subspace of dimension $n - 1$: a line in $\mathbb{R}^2$, a plane in $\mathbb{R}^3$.

Ex (p. 264): $\{v \in \mathbb{R}^n : u \cdot v = 0\}$ is a subspace of dimension ?

Answer: dimension $n - 1$ (when $u \neq 0$)

The notes' proof: $T(v) = u \cdot v$ is linear and onto ($T(u) \neq 0$), the set is $\operatorname{Ker} T$, and $\dim \operatorname{Ker} T = n - \dim \operatorname{Im} T = n - 1$.

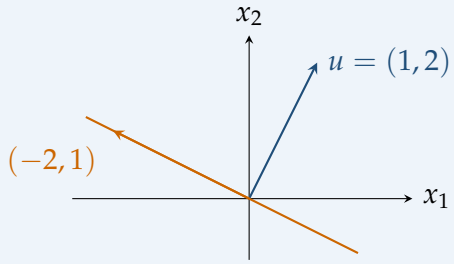
- *Linear*, $T : \mathbb{R}^n \to \mathbb{R}$: by linearity in the second slot (p. 259).
- *Onto* (needs $u \neq 0$): $T(u) = \|u\|^2 \neq 0$, so the image is a nonzero subspace of $\mathbb{R}$, hence all of $\mathbb{R}$. Explicitly, for any c in $\mathbb{R}$, $T\left(\frac{c}{\|u\|^2}u\right) = \frac{c}{\|u\|^2}\,u \cdot u = c$.
- So $\operatorname{Im} T = \mathbb{R}$ has dimension 1, and by the Rank Theorem (dimension of kernel + dimension of image = dimension of domain, Lay §4.6) $\dim \operatorname{Ker} T = n - 1$. A kernel is always a subspace.

If $u = 0$, every v qualifies and the set is all of $\mathbb{R}^n$ (dimension n), which is why the notes require $u \neq 0$.

Ex (p. 264): $u = (1, 2)$

Fill-in: $\{v \in \mathbb{R}^2 : v \cdot u = 0\} = \text{span}\{(-2, 1)\}$

$v \cdot u = v_1 + 2v_2 = 0$ means $v_1 = -2v_2$, with v_2 free: $v = v_2(-2, 1)$. A line through 0, perpendicular to u, and of dimension $1 = n - 1$. ✓
Check: $(1, 2) \cdot (-2, 1) = -2 + 2 = 0$. ✓



8 Orthogonal complements (pp. 265–267)

$W^\perp$ is everything orthogonal to all of W; to test membership, it is enough to check a basis of W.

Definition (orthogonal complement, p. 265)

Let W be a subspace of $\mathbb{R}^n$. The **orthogonal complement** of W is $W^\perp = \{v \in \mathbb{R}^n : v \cdot w = 0 \text{ for every } w \text{ in } W\}$.

Why is $W^\perp$ a subspace? It contains 0 and is closed under linear combinations

- $0 \cdot w = 0$ for every w (p. 258), so $0 \in W^\perp$.
- If $v_1, v_2 \in W^\perp$ and c_1, c_2 are scalars, then for every w in W, $(c_1v_1 + c_2v_2) \cdot w = c_1(v_1 \cdot w) + c_2(v_2 \cdot w) = 0 + 0 = 0$ by linearity (p. 259). So $c_1v_1 + c_2v_2 \in W^\perp$.

(Another proof comes on p. 267: $W^\perp$ is a null space.)

Fill-in: $W = \text{span}\{e_1, e_3\}$ **in** $\mathbb{R}^3 \Rightarrow W^\perp = \text{span}\{e_2\}$

By the "key point" on p. 266, $v \in W^\perp \iff v \cdot e_1 = 0$ and $v \cdot e_3 = 0$, i.e. $v_1 = 0$ and $v_3 = 0$. So $v = v_2e_2$: $W^\perp$ is the x_2-axis, perpendicular to the x_1x_3-plane W. *Check:* $\dim W + \dim W^\perp = 2 + 1 = 3$ (p. 266). ✓

Fill-in: $\{0\}^\perp = \mathbb{R}^n$ **and** $(\mathbb{R}^n)^\perp = \{0\}$

$\{0\}^\perp$: every v satisfies $v \cdot 0 = 0$, so $\{0\}^\perp = \mathbb{R}^n$.
$(\mathbb{R}^n)^\perp$: if $v \cdot w = 0$ for every w in $\mathbb{R}^n$, then $v = 0$ by non-degeneracy (p. 262). And 0 does qualify. So $(\mathbb{R}^n)^\perp = \{0\}$.
Check: dimensions $0 + n = n$ and $n + 0 = n$. ✓

Theorem (p. 266)

If W is a subspace of $\mathbb{R}^n$, then $\dim W + \dim W^\perp = n$.

Proof (pp. 266–267). If $W = \{0\}$, then $W^\perp = \mathbb{R}^n$, so $0 + n = n$. If $W \neq \{0\}$, let $v_1, \ldots, v_k$ be a basis of W. The key point is

$$x \in W^\perp \iff x \cdot v_i = 0 \text{ for all } i = 1, \ldots, k.$$

Indeed, "$\Rightarrow$" holds by the definition of $W^\perp$ (each v_i is in W).

"$\Leftarrow$": take any w in W. Why is $x \cdot w = 0$? Expand w in the basis

The notes: $w = c_1 v_1 + \cdots + c_k v_k$, so $x \cdot w = c_1\, x \cdot v_1 + \cdots + c_k\, x \cdot v_k$ with every $x \cdot v_i = 0$. In detail: since $v_1, \ldots, v_k$ is a basis of W, write $w = a_1 v_1 + \cdots + a_k v_k$. By linearity (p. 259),

$$x \cdot w = a_1(x \cdot v_1) + \cdots + a_k(x \cdot v_k) = a_1 \cdot 0 + \cdots + a_k \cdot 0 = 0.$$

So x is orthogonal to every w in W: $x \in W^\perp$. (The notes write "take any w in $W^\perp$"; it should be w in W, see the corrections.)

Matrix form (p. 267). Write the system $x \cdot v_1 = 0, \ \ldots, \ x \cdot v_k = 0$ in matrix form. Since $x \cdot v_i = v_i^T x$, the system is

$$\begin{bmatrix} v_1^T \\ \vdots \\ v_k^T \end{bmatrix} x = 0, \qquad \text{i.e.} \quad Ax = 0 \ \text{ with } \ A = [v_1 \ \cdots \ v_k]^T \ (k \times n).$$

So $W^\perp = \operatorname{Nul} A$, and

Fill-in: $\dim W^\perp = n - \operatorname{rank} A = n - \operatorname{rank} A^T = n - k = n - \dim W$

- $\dim \operatorname{Nul} A = n - \operatorname{rank} A$ by the Rank Theorem (Lay §2.9; A has n columns).
- $\operatorname{rank} A = \operatorname{rank} A^T$ (Lay §4.6).
- $A^T = [v_1 \ \cdots \ v_k]$ has independent columns (a basis of W), so $\operatorname{rank} A^T = k = \dim W$.

Hence $\dim W^\perp = n - \dim W$, i.e. $\dim W + \dim W^\perp = n$. ■

Note (p. 267): if v is in W and in $W^\perp$, then $v = 0$

The notes: $0 = v \cdot v$. That is, $v \in W^\perp$ means v is orthogonal to everything in W, including v itself: $v \cdot v = 0$. By positivity, $v = 0$. So $W \cap W^\perp = \{0\}$.

9 A plane through the origin; the row space (p. 268)

The orthogonal complement of one nonzero vector in $\mathbb{R}^3$ is a plane, and its equation is read off from the vector.

Ex (p. 268): $v = (1, 2, -1)$

Then $\text{span}\{v\}^\perp = \{x \in \mathbb{R}^3 : x \cdot v = 0\}$, $x = (x_1, x_2, x_3)$, is a plane in $\mathbb{R}^3$ (dimension $3 - 1 = 2$ by p. 266).

What is the equation of the plane? $x_1 + 2x_2 - x_3 = 0$

$x \cdot v = 1 \cdot x_1 + 2 \cdot x_2 + (-1) \cdot x_3$, so the plane is $x_1 + 2x_2 - x_3 = 0$. The coefficients of the equation are the entries of v: v is a *normal vector* of the plane.
Check: solving for $x_1 = -2x_2 + x_3$ gives the basis $(-2, 1, 0)$, $(1, 0, 1)$; both have dot product 0 with v ($-2 + 2 + 0 = 0$, $1 + 0 - 1 = 0$). ✓

Definition (recall, p. 268)

The **row space** of A is $\text{Row } A = \text{Col}(A^T)$ = the span of the rows of A (Lay §4.6).

If A is $m \times n$, is $\text{Row } A$ in $\mathbb{R}^n$ or $\mathbb{R}^m$? In $\mathbb{R}^n$

Each row of an $m \times n$ matrix has n entries (one per column), so the rows, and their span, live in $\mathbb{R}^n$. Equivalently, A^T is $n \times m$, so $\text{Col}(A^T)$ is in $\mathbb{R}^n$. (By contrast $\text{Col } A$ is in $\mathbb{R}^m$.)

10 $(\text{Row } A)^\perp = \text{Nul } A$ and the four fundamental subspaces (pp. 269–270)

A vector solves $Ax = 0$ exactly when it is orthogonal to every row of A.

Theorem (p. 269)

$(\text{Row } A)^\perp = \text{Nul } A$ and $(\text{Col } A)^\perp = \text{Nul}(A^T)$.

The two statements are equivalent. Why? Replace A by A^T

The notes: "Replace A by A^T." Suppose the first statement holds for every matrix. Apply it to A^T: $(\text{Row } A^T)^\perp = \text{Nul}(A^T)$. Now $\text{Row } A^T = \text{Col}\left((A^T)^T\right) = \text{Col } A$, so this says $(\text{Col } A)^\perp = \text{Nul}(A^T)$, the second statement. Applying the second to A^T gives back the first in the same way.

Proof of $(\text{Row } A)^\perp = \text{Nul } A$ (pp. 269–270). If A is $m \times n$, then

$$\dim \text{Row } A = \dim \text{Col } A^T = \text{rank } A^T = \text{rank } A = n - \dim \text{Nul } A,$$

and by the theorem on p. 266, $\dim(\text{Row } A)^\perp = n - \dim \text{Row } A = \dim \text{Nul } A$. So $\text{Nul } A$ and $(\text{Row } A)^\perp$ have the same dimension.

If x is in $\operatorname{Nul} A$, is $x \perp$ each row of A? Yes

Write the rows of A as $r_1, \ldots, r_m$ (as vectors in $\mathbb{R}^n$). Row-by-row multiplication gives

$$A = \begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix}, \qquad Ax = \begin{bmatrix} r_1 \cdot x \\ \vdots \\ r_m \cdot x \end{bmatrix}.$$

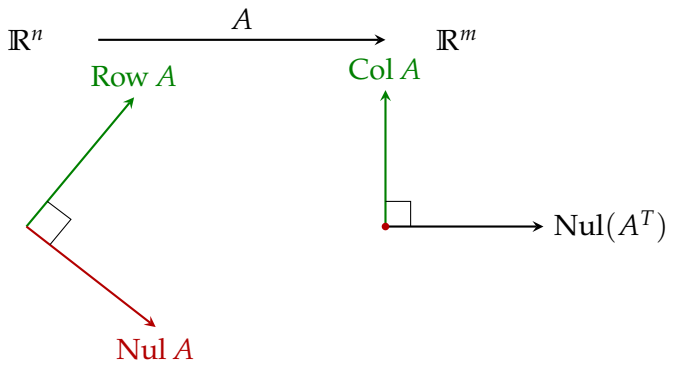
So $Ax = 0 \iff r_i \cdot x = 0$ for every i. By the key point on p. 266 (the rows span Row A), x is then orthogonal to all of Row A. So $\operatorname{Nul} A \subseteq (\operatorname{Row} A)^\perp$ (p. 270).

Did we show $\operatorname{Nul} A = (\operatorname{Row} A)^\perp$? Yes

The notes: "They have the same dimension. So yes, we did." We have $\operatorname{Nul} A \subseteq (\operatorname{Row} A)^\perp$, and both have the same dimension. A subspace contained in another subspace of the same finite dimension is equal to it (a basis of the smaller one is an independent set of the right size in the bigger one, hence a basis of it, Lay §4.5). So $\operatorname{Nul} A = (\operatorname{Row} A)^\perp$. ■
(In fact the argument above shows both inclusions directly: $Ax = 0 \iff x \perp$ every row $\iff x \in (\operatorname{Row} A)^\perp$, using the key point in both directions. The dimension count is a useful double check.)

Four fundamental subspaces (p. 270)

An $m \times n$ matrix A gives a map $\mathbb{R}^n \to \mathbb{R}^m$. In the domain $\mathbb{R}^n$ sit $\operatorname{Row} A$ and $\operatorname{Nul} A$; in the codomain $\mathbb{R}^m$ sit $\operatorname{Col} A$ and $\operatorname{Nul}(A^T)$. By the theorem, each pair is orthogonal complements, so the dimensions in each space add up: $\operatorname{rank} A + \dim \operatorname{Nul} A = n$ and $\operatorname{rank} A + \dim \operatorname{Nul}(A^T) = m$.



The picture shows $\operatorname{Row} A \perp \operatorname{Nul} A$ inside $\mathbb{R}^n$ and $\operatorname{Col} A \perp \operatorname{Nul}(A^T)$ inside $\mathbb{R}^m$. (In the notes the red dot marks 0 in $\mathbb{R}^m$: A sends $\operatorname{Nul} A$ to it.)

11 Benefits of orthonormality (pp. 271–272)

Orthonormal vectors are automatically independent, and the coordinates of a vector in an orthonormal basis are just dot products, with no system to solve.

Theorem (p. 271)

If $\{u_1, \ldots, u_k\}$ is orthonormal in $\mathbb{R}^n$, then $\{u_1, \ldots, u_k\}$ is independent.

Proof, blank filled in: $0 \cdot u_1 = c_1$, so $c_1 = 0$

Suppose $c_1u_1 + \cdots + c_ku_k = 0$. Dot both sides with u_1:

$$\begin{aligned} 0 = 0 \cdot u_1 &= (c_1u_1 + \cdots + c_ku_k) \cdot u_1 = c_1(u_1 \cdot u_1) + c_2(u_2 \cdot u_1) + \cdots + c_k(u_k \cdot u_1) \\ &= c_1 \cdot 1 + 0 + \cdots + 0 = c_1. \end{aligned}$$

So $c_1 = 0$. Dotting with u_i instead gives $c_i = 0$ for every i. So the only combination giving 0 is the trivial one: the set is independent. ■

Theorem (p. 271)

If $\mathcal{B} = \{u_1, \ldots, u_k\}$ is an orthonormal basis of a subspace W of $\mathbb{R}^n$ and x is in W, then

$$x = (x \cdot u_1)u_1 + \cdots + (x \cdot u_k)u_k, \qquad \text{that is,} \qquad [x]_{\mathcal{B}} = \begin{bmatrix} x \cdot u_1 \\ \vdots \\ x \cdot u_k \end{bmatrix}.$$

Proof, blank filled in: $c_i = x \cdot u_i$

Since $\mathcal{B}$ is a basis of W and $x \in W$, $x = c_1u_1 + \cdots + c_ku_k$ for some scalars. Dot with u_i; as in the previous proof, every term except the i-th vanishes:

$$x \cdot u_i = c_1(u_1 \cdot u_i) + \cdots + c_k(u_k \cdot u_i) = c_i(u_i \cdot u_i) = c_i.$$

So $c_i = x \cdot u_i$. ■

Example (addition): $\mathcal{B} = \{u_1, u_2\} = \left\{\frac{1}{\sqrt{2}}(1,1), \frac{1}{\sqrt{2}}(1,-1)\right\}$ is an orthonormal basis of $\mathbb{R}^2$, and for $x = (3,1)$: $x \cdot u_1 = 4/\sqrt{2}$, $x \cdot u_2 = 2/\sqrt{2}$. *Check:* $\frac{4}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}(1,1) + \frac{2}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}(1,-1) = (2,2) + (1,-1) = (3,1)$. ✓

Definition (orthogonal basis, p. 272)

An **orthogonal basis** of W is a basis $\{v_1, \ldots, v_k\}$ of W such that $v_i \cdot v_j = 0$ if $i \neq j$.

Then $\left\{\frac{v_1}{\|v_1\|}, \ldots, \frac{v_k}{\|v_k\|}\right\}$ is an orthonormal basis of W.

Why? Normalizing keeps the dot products 0 and makes the lengths 1

Basis vectors are nonzero, so each $\|v_i\| \neq 0$ and $u_i = v_i/\|v_i\|$ makes sense.

- *Length* 1: by the note on p. 261.
- *Orthogonal:* for $i \neq j$, $u_i \cdot u_j = \dfrac{v_i \cdot v_j}{\|v_i\|\,\|v_j\|} = 0$ by linearity.
- *Basis:* the u_i are k orthonormal vectors in W, hence independent (p. 271); $k = \dim W$ independent vectors in W form a basis (Lay §4.5). (Or: each u_i is a nonzero multiple of v_i, so they span the same space.)

Example: $(1,1)$, $(1,-1)$ is an orthogonal basis of $\mathbb{R}^2$; normalizing gives the orthonormal basis in the example above. ✓

Summary

Fact	Formula
Dot product, length	$u \cdot v = u^T v = \sum u_i v_i$, $\|u\| = \sqrt{u \cdot u}$
Expansion	$\|u \pm v\|^2 = \|u\|^2 \pm 2\,u \cdot v + \|v\|^2$
Scaling, unit vector	$\|cu\| = \|c\|\|u\|$; $u/\|u\|$ has length 1
Angle	$\cos\theta = \dfrac{u \cdot v}{\|u\|\|v\|}$, $0 \leq \theta \leq \pi$
Pythagoras	$u \perp v \iff \|u \pm v\|^2 = \|u\|^2 + \|v\|^2$
Orthogonal complement	$x \in W^\perp \iff x \cdot v_i = 0$ for a basis v_i of W; $\dim W + \dim W^\perp = n$
Fundamental subspaces	$(\operatorname{Row} A)^\perp = \operatorname{Nul} A$, $(\operatorname{Col} A)^\perp = \operatorname{Nul}(A^T)$
Orthonormal sets	independent; $[x]_\mathcal{B} = (x \cdot u_1, \ldots, x \cdot u_k)$

Corrections to the handwritten notes

Page	As written	Should be	Why
266	"take any w in $W^\perp$"	"take any w in W"	we must show x is orthogonal to W
268	"a plane in $\mathbb{R}^n$"	"a plane in $\mathbb{R}^3$"	v and x are in $\mathbb{R}^3$