

Linear Algebra Notes — Lecture 9/1

Homogeneous Systems and Linear Independence

Lecture pages 22–31 · Lay §1.5, §1.7

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Overview

Last time (8/27 notes, pp. 9–21) we learned to solve $Ax = b$ by row reduction, wrote Ax as a linear combination of the columns of A , and saw that $Ax = b$ is consistent exactly when b is in the span of the columns. This lecture looks at the special case $b = 0$ and uses it to answer a new question: when does a list of vectors contain no redundant vectors?

1. **Homogeneous systems.** $AX = 0$ always has the solution $X = 0$. It has a nonzero solution exactly when some column of A has no pivot.
2. **Linear independence.** Vectors $v_1, \dots, v_k$ are independent if the only way to combine them into 0 is with all weights 0. Dependent means one of them is a combination of the others.

3. **The test.** Put the vectors as the columns of a matrix. They are independent exactly when every column is pivotal. In particular, more than n vectors in $\mathbb{R}^n$ are always dependent.

Every blank and question in the notes is answered in a blue box. Typos are listed at the end.

Notation used throughout.

- $A = [a_1 \cdots a_n]$: the $m \times n$ matrix with columns $a_1, \dots, a_n \in \mathbb{R}^m$.
- $\text{RREF}(A)$: the reduced row echelon form of A (8/25 notes, p. 7). Often written R .
- $AX = x_1 a_1 + \cdots + x_n a_n$ for $X = (x_1, \dots, x_n)$: the matrix–vector product (8/27 notes, p. 18).
- $\text{span}\{v_1, \dots, v_k\}$: all linear combinations $c_1 v_1 + \cdots + c_k v_k$ (8/27 notes, p. 15).

Recipe: is $\{v_1, \dots, v_k\}$ independent?

1. Form the matrix $A = [v_1 \cdots v_k]$.
2. Row reduce (an echelon form is enough to see where the pivots are).
3. Independent $\iff$ every column has a pivot. If some column has no pivot, the vectors are dependent, and solving $AX = 0$ gives an explicit dependence relation.

1 Homogeneous systems (pp. 22–23)

A homogeneous system $AX = 0$ is never inconsistent, and its solutions are closed under scaling; it has nonzero solutions exactly when A has a column without a pivot.

Definition (p. 22)

A linear system $AX = 0$ (right-hand side equal to the zero vector) is called *homogeneous*.

Key property (p. 22). For any number c : if $AX = 0$, then $A(cX) = 0$.

Why is cX again a solution?

Write $A = [a_1 \cdots a_n]$ and $X = (x_1, \dots, x_n)$. Then $cX = (cx_1, \dots, cx_n)$, and by the definition of the matrix–vector product (8/27 notes, p. 18),

$$\begin{aligned} A(cX) &= (cx_1)a_1 + \cdots + (cx_n)a_n && \text{(definition of } A \cdot \text{vector)} \\ &= c(x_1 a_1 + \cdots + x_n a_n) && \text{(factor out } c) \\ &= c(AX) = c \cdot 0 = 0. && \text{(definition again, then } AX = 0) \end{aligned}$$

Check: $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $X = (2, -1)$: $AX = (0, 0)$, and $A(3X) = A(6, -3) = (6 - 6, 18 - 18) = (0, 0)$. ✓

Example (p. 22). $AX = 0$ has an obvious solution, namely $X = 0$ (the *trivial* solution): $A0 = 0a_1 + \cdots + 0a_n = 0$. So a homogeneous system is always consistent. The interesting question is

whether it has other solutions.

Theorem (p. 22)

For an $m \times n$ matrix A the following are equivalent:

1. $AX = 0$ has a nonzero solution;
2. $AX = 0$ has infinitely many solutions;
3. (when the system is solved by row reduction) there is a free variable;
4. A has a column with no pivot.

Convention (p. 23)

Row operations do not move columns around. So we say a column of A is *pivotal* if the same column of $\text{RREF}(A)$ has a pivot.

The notes give the proof as “ $2 \Leftrightarrow 3 \Leftrightarrow 4$ trivial”, “ $2 \Rightarrow 1$ ” (the arrow), and a short argument for $1 \Rightarrow 2$. Here are all the steps.

Proof (the “trivial” steps filled in)

$3 \Leftrightarrow 4$. Row reduce the augmented matrix $[A \mid 0]$. Row operations never change the zero last column, so $\text{RREF}[A \mid 0] = [\text{RREF}(A) \mid 0]$. A variable x_j is free exactly when column j of $\text{RREF}(A)$ has no pivot, i.e. when column j of A is not pivotal. So “there is a free variable” and “some column has no pivot” say the same thing.

$2 \Leftrightarrow 4$. By the solution theorem from last time (8/27 notes, p. 10), a consistent system has a unique solution exactly when each column of $R = \text{RREF}(A)$ is pivotal, and infinitely many solutions exactly when some column of R is not pivotal. $AX = 0$ is consistent (it has $X = 0$), so these are the only two cases. Hence “infinitely many” $\Leftrightarrow$ “some column has no pivot”.

$2 \Rightarrow 1$. Infinitely many solutions cannot all be the zero vector, so at least one is nonzero.

$1 \Rightarrow 2$ (the notes’ argument). If $X \neq 0$ solves $AX = 0$, then so does cX for every number c by the key property. Different c give different vectors (if $cX = c'X$ then $(c - c')X = 0$, and some entry of X is nonzero, so $c = c'$). So there are infinitely many solutions.

Together: $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4$. ■

Example (p. 23): $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

Subtracting $3 \times$ row 1 from row 2 (the notes’ “ $-3 \boxed{1}$ ”):

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

This is already $\text{RREF}(A)$. There is no pivot in column 2, so by the theorem $AX = 0$ has a nonzero solution and infinitely many solutions.

Fill-in: a nonzero solution is $X = (2, -1)$; all solutions are $x_2(-2, 1)$

The reduced system is $x_1 + 2x_2 = 0$ with x_2 free, so $x_1 = -2x_2$ and

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad x_2 \in \mathbb{R}.$$

The notes' choice $X = (2, -1)$ is the case $x_2 = -1$. Every multiple $c(2, -1)$ is also a solution: infinitely many.

Check: $A(2, -1)^T = (1 \cdot 2 + 2 \cdot (-1), 3 \cdot 2 + 6 \cdot (-1))^T = (0, 0)^T$. ✓

2 Linear independence (p. 24)

A list of vectors is independent when no vector in it is wasted: the only combination giving 0 is the one with all weights 0.

Motivation (p. 24). If $v_1, \dots, v_k$ are vectors in $\mathbb{R}^m$, how do we describe $\text{span}\{v_1, \dots, v_k\}$ with the least amount of data?

Example (p. 24). $\text{span}\{v_1, v_2, v_1 + v_2\} = \text{span}\{v_1, v_2\}$. The notes illustrate with one combination: $5v_1 + 2v_2 - 3(v_1 + v_2) = 2v_1 - v_2$.

Why is $\text{span}\{v_1, v_2, v_1 + v_2\} = \text{span}\{v_1, v_2\}$?

Show each set is inside the other.

$\supseteq$: a combination $av_1 + bv_2$ is also a combination of $v_1, v_2, v_1 + v_2$ (use weight 0 on $v_1 + v_2$).

$\subseteq$: any combination of the three vectors can be regrouped:

$$av_1 + bv_2 + c(v_1 + v_2) = (a + c)v_1 + (b + c)v_2,$$

which is a combination of v_1, v_2 alone. The notes' example is the case $a = 5, b = 2, c = -3$: $(5 - 3)v_1 + (2 - 3)v_2 = 2v_1 - v_2$. ✓

Meaning: the vector $v_1 + v_2$ adds nothing new to the span, so we can throw it away.

Idea (p. 24). Remove those vectors in $\{v_1, \dots, v_k\}$ that do not contribute to the span. The definition below detects when there is something to remove.

Definition (linearly independent, p. 24)

Vectors $\{v_1, \dots, v_k\}$ in $\mathbb{R}^m$ are (linearly) independent if the vector equation

$$c_1v_1 + \dots + c_kv_k = 0$$

with unknowns $c_1, \dots, c_k$ has only the zero solution $(c_1, \dots, c_k) = (0, \dots, 0)$.

$\{v_1, \dots, v_k\}$ is (linearly) dependent if it isn't independent, i.e. there are weights $c_1, \dots, c_k$, not all zero, with $c_1v_1 + \dots + c_kv_k = 0$. Such an equation is called a *dependence relation*.

In the example, $1 \cdot v_1 + 1 \cdot v_2 + (-1)(v_1 + v_2) = 0$ is a dependence relation, so $\{v_1, v_2, v_1 + v_2\}$ is dependent.

3 Dependent means one vector is a combination of the others (p. 25)

Dependence says exactly that some vector in the list is redundant: it is a linear combination of the rest.

Theorem (p. 25)

$\{v_1, \dots, v_k\}$ is dependent $\iff$ for some $i = 1, \dots, k$, the vector v_i is a linear combination of $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_k$.

Proof (both directions written out)

$\Rightarrow$. By definition of dependent we have $c_1v_1 + \dots + c_kv_k = 0$ with some $c_i \neq 0$. Move every other term to the right side and divide by c_i (allowed because $c_i \neq 0$; this is the step the notes' arrow points to):

$$v_i = -\frac{1}{c_i}(c_1v_1 + \dots + c_{i-1}v_{i-1} + c_{i+1}v_{i+1} + \dots + c_kv_k).$$

This writes v_i as a combination of the other vectors.

$\Leftarrow$. Given $v_i = a_1v_1 + \dots + a_{i-1}v_{i-1} + a_{i+1}v_{i+1} + \dots + a_kv_k$, rewrite it as

$$a_1v_1 + \dots + a_{i-1}v_{i-1} + (-1)v_i + a_{i+1}v_{i+1} + \dots + a_kv_k = 0.$$

This is an equation $c_1v_1 + \dots + c_kv_k = 0$ with $c_i = -1 \neq 0$ (the notes: "Note $c_i \neq 0$ "). So the vectors are dependent. ■

Fine print. The theorem says *some* v_i is a combination of the others, not *every* one. Example: $\{(1,0), (2,0), (0,1)\}$ is dependent because $(2,0) = 2(1,0) + 0(0,1)$, but $(0,1)$ is not a combination of $(1,0)$ and $(2,0)$ (their second entries are 0).

Example (p. 25): find all h such that the columns of $\begin{bmatrix} 1 & 2 \\ 3 & h \end{bmatrix}$ are dependent

Answer: $h = 6$

By the Note below, two vectors are dependent exactly when they are proportional. The first column $(1,3)$ is not a multiple of $(2,h)$ unless $(2,h) = c(1,3)$: the first entry forces $c = 2$, and then $h = 3c = 6$. (The other option, $(1,3) = c(2,h)$, gives $c = \frac{1}{2}$ and again $h = 6$.) The notes: $\begin{bmatrix} 2 \\ 6 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, so v_1, v_2 are proportional.

Check by row reduction: $\begin{bmatrix} 1 & 2 \\ 3 & h \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & 2 \\ 0 & h-6 \end{bmatrix}$. Column 2 has a pivot unless $h - 6 = 0$. So the columns fail to be all pivotal exactly when $h = 6$ (test of Section 4). ✓

Note (p. 25). $\{v_1, v_2\}$ is dependent $\iff v_1 = cv_2$ or $v_2 = cv_1$ for some number c .

Why? It is the theorem with $k = 2$

With two vectors, “ v_i is a combination of the other vectors” means v_1 is a multiple of v_2 ($i = 1$) or v_2 is a multiple of v_1 ($i = 2$), since a combination of one vector is just a multiple of it.

Why both options are needed: take $v_1 = (1, 0)$, $v_2 = (0, 0)$. Then $v_2 = 0 \cdot v_1$, so the set is dependent, but v_1 is not a multiple of v_2 (every multiple of 0 is 0).

4 Pivotal columns are independent (pp. 26–28)

The pivotal columns of any matrix form an independent set, and a matrix has independent columns exactly when all its columns are pivotal.

Example (p. 26): the pivotal columns of an RREF are independent

Take the RREF matrix

$$R = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Its pivots (the leading 1s) are in columns 1, 3 and 5. Those columns are the standard vectors $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$, because in an RREF each pivot is a 1 and is the only nonzero entry in its column (8/25 notes, p. 7). Call the weights c_1, c_2, c_3 .

Fill-in: $c_1 = c_2 = c_3 = 0$, so the pivotal columns are independent

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies c_1 = c_2 = c_3 = 0.$$

The same works for any RREF: the pivotal columns are $e_1, e_2, \dots, e_r$ (in order, since each pivot is in a lower row than the one before), and $c_1 e_1 + \dots + c_r e_r$ is the vector whose first r entries are $c_1, \dots, c_r$. It is 0 only if all $c_j = 0$.

Example (p. 26): a list containing the zero vector is dependent

If some $v_i = 0$, then $\{v_1, \dots, v_k\}$ is dependent.

Why? Put weight 1 on the zero vector

For $i = 1$ (the notes' case):

$$1 \cdot v_1 + 0v_2 + 0v_3 + \dots + 0v_k = 1 \cdot 0 + 0 = 0,$$

and the weight $c_1 = 1$ is not zero, so this is a dependence relation. For general i , take $c_i = 1$ and all other $c_j = 0$: then $c_1 v_1 + \dots + c_k v_k = 1 \cdot v_i = 0$.

Theorem (p. 27)

Let $A = [v_1 \cdots v_n]$ be an $m \times n$ matrix, where v_i is column i of A . Then

$$\{v_1, \dots, v_n\} \text{ is independent} \iff \text{each column of } A \text{ is pivotal}$$

(that is, $\text{RREF}(A)$ has n pivots).

Proof (with the cited fact)

By the definition of the matrix–vector product (8/27 notes, p. 18),

$$c_1 v_1 + \cdots + c_n v_n = 0 \iff A \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = 0.$$

So the vectors are independent exactly when this homogeneous system has only the zero solution. By the theorem on p. 22 (statement 1 $\iff$ statement 4, negated), $AX = 0$ has only the zero solution $\iff$ no column of A is without a pivot $\iff$ each column of A is pivotal. Each pivotal column holds exactly one pivot, so this is the same as “ $\text{RREF}(A)$ has n pivots”. ■

More generally:

Theorem (p. 27)

The pivotal columns of any matrix are independent.

Proof (pp. 27–28). Suppose $A = [v_1 \cdots v_n]$ and $v_{i_1}, \dots, v_{i_k}$ are its pivotal columns ($i_1 < \cdots < i_k$). To show they are independent, suppose

$$c_{i_1} v_{i_1} + \cdots + c_{i_k} v_{i_k} = 0.$$

We must show every $c_{i_j} = 0$. Let $C = (c_1, \dots, c_n)$, where $c_i = 0$ if i is not among $i_1, \dots, i_k$. Then the non-pivotal columns get weight 0, so

$$AC = c_{i_1} v_{i_1} + \cdots + c_{i_k} v_{i_k} = 0.$$

The notes ask “ $\Rightarrow C = 0??$ ”.

Why does $AC = 0$ force $C = 0$? Because it does for $R = \text{RREF}(A)$

Step 1. $AC = 0 \iff RC = 0$. Row operations do not change the solutions of a system (8/25 notes, p. 6), and they leave the zero right-hand side zero. So the systems $AX = 0$ and $RX = 0$ have the same solutions; in particular $AC = 0 \iff RC = 0$.

Step 2. Write $R = [r_1 \cdots r_n]$. Since $c_i = 0$ for the non-pivotal columns,

$$0 = RC = c_1 r_1 + \cdots + c_n r_n = c_{i_1} r_{i_1} + \cdots + c_{i_k} r_{i_k}.$$

Step 3. The columns $r_{i_1}, \dots, r_{i_k}$ are the pivotal columns of an RREF, and these are independent by the example on p. 26 (they are $e_1, \dots, e_k$). So $c_{i_1}, \dots, c_{i_k}$ must all be zero, i.e. $C = 0$. ■

Check on the p. 26 matrix: R above has pivotal columns 1, 3, 5; the 3×3 matrix formed by them is the identity, which has 3 pivots, so they are independent. ✓

Pivotal columns of A , not of R . The theorem is about the columns of the *original* matrix A in the pivot positions. Row operations change the columns, so the pivotal columns of A are usually not $e_1, \dots, e_k$; the proof only uses R as a tool, through Step 1.

5 The independence test and an example with a parameter (pp. 28–29)

To decide independence, row reduce the matrix whose columns are the vectors and count pivots.

Conclusion (p. 28)

To test if vectors are independent, put them as columns in a matrix and check if all columns are pivotal.

Example (pp. 28–29): find all h such that $(1, 1, h)$, $(1, h, 1)$, $(h, 1, 1)$ are dependent

Put the three vectors as the columns of a matrix and row reduce. First subtract row 1 from row 2 and h times row 1 from row 3 (the notes' " -1 " and " $-h1$ "); then add row 2 to row 3 (" $+2$ "):

$$\begin{bmatrix} 1 & 1 & h \\ 1 & h & 1 \\ h & 1 & 1 \end{bmatrix} \xrightarrow[R_3 - hR_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & h \\ 0 & h-1 & 1-h \\ 0 & 1-h & 1-h^2 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & 1 & h \\ 0 & h-1 & 1-h \\ 0 & 0 & 2-h-h^2 \end{bmatrix}.$$

(Row 3 after the first step: $h - h \cdot 1 = 0$, $1 - h \cdot 1 = 1 - h$, $1 - h \cdot h = 1 - h^2$. After the second: $(1 - h^2) + (1 - h) = 2 - h - h^2$.) The last entry factors as

$$2 - h - h^2 = -(h^2 + h - 2) = -(h - 1)(h + 2).$$

Answer: dependent exactly when $h = 1$ or $h = -2$

If $h \neq 1$ and $h \neq -2$: both $h - 1$ and $2 - h - h^2 = -(h - 1)(h + 2)$ are nonzero, so the echelon form has a pivot in every column. (Scaling and clearing reduce it to the identity I_3 , as the notes draw.) All columns are pivotal, so the vectors are independent.

If $h = 1$: the matrix becomes $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$: only column 1 is pivotal, so the vectors are dependent. Indeed all three vectors equal $(1, 1, 1)$. Check: $(1, 1, 1) - (1, 1, 1) + 0 \cdot (1, 1, 1) = 0$. ✓

If $h = -2$: the matrix becomes $\begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$: no pivot in column 3, so the vectors are dependent. Solving $AC = 0$: c_3 free, $-3c_2 + 3c_3 = 0$ gives $c_2 = c_3$, and $c_1 + c_2 - 2c_3 = 0$ gives $c_1 = c_3$. With $c_3 = 1$: Check: $(1, 1, -2) + (1, -2, 1) + (-2, 1, 1) = (0, 0, 0)$, i.e. $1 \cdot v_1 + 1 \cdot v_2 + 1 \cdot v_3 = 0$. ✓

Watch the division. At $h = 1$ the entry $h - 1$ in row 2 is 0, so the second pivot is not where it is for other h . That is why the notes treat $h = 1$ separately (the underlined $h - 1$) instead of just reading off the last entry. Whenever an entry with a parameter is used as a pivot, the values that make it zero need their own case.

6 One vector, and too many vectors (p. 29)

A single vector is independent exactly when it is nonzero, and more than n vectors in $\mathbb{R}^n$ are always dependent.

Example (p. 29): if v is in $\mathbb{R}^n$, is $\{v\}$ dependent?

The question is whether $cv = 0$ forces $c = 0$ (the notes' " $\Rightarrow c = 0$??").

Answer: $\{v\}$ is independent if $v \neq 0$, dependent if $v = 0$

The notes: "True $\iff v \neq 0$."

If $v \neq 0$: some entry $v_j \neq 0$. The j -th entry of $cv = 0$ says $cv_j = 0$, so $c = 0$. Only the zero weight works: independent.

If $v = 0$: $1 \cdot v = 0$ with weight $1 \neq 0$: dependent (the zero-vector example of p. 26).

Check with the test: the $n \times 1$ matrix $[v]$ has a pivot in its one column iff some entry is nonzero. ✓

Theorem (p. 29)

If $v_1, \dots, v_k$ are in $\mathbb{R}^n$ and $k > n$, then $\{v_1, \dots, v_k\}$ is dependent.

Proof (expanded)

Let $A = [v_1 \cdots v_k]$; it is $n \times k$ (n rows, k columns). If $\{v_1, \dots, v_k\}$ were independent, all k columns of A would be pivotal (theorem on p. 27), so $\text{RREF}(A)$ would have k pivots. But each row of an RREF contains at most one pivot (the pivot is the leftmost nonzero entry of its row), so there are at most n pivots. k pivots cannot fit in $n < k$ rows: contradiction. So the set is dependent. ■

Worked example (addition). The three vectors $(1,3)$, $(2,1)$, $(0,5)$ in $\mathbb{R}^2$ must be dependent. Row reduction of $\begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 5 \end{bmatrix}$ gives $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$, so column 3 is not pivotal, and $c_3 = 1$, $c_1 = -2$, $c_2 = 1$ solves $AC = 0$. Check: $-2(1,3) + (2,1) + (0,5) = (-2+2+0, -6+1+5) = (0,0)$. ✓

The converse fails. $k \leq n$ does *not* make vectors independent: $(1,2)$ and $(2,4)$ are only two vectors in $\mathbb{R}^2$ but are dependent. The theorem only tells you about the case $k > n$.

7 A common mistake (p. 30)

Checking the vectors two at a time is not enough: three vectors can be pairwise independent and still dependent.

Example (p. 30). Suppose v_1, v_2, v_3 are in $\mathbb{R}^3$ and $\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}$ are independent. Can we conclude $\{v_1, v_2, v_3\}$ is independent?

Answer: No. Counterexample $v_1 = (1, 0, 0), v_2 = (0, 1, 0), v_3 = (1, 1, 0)$

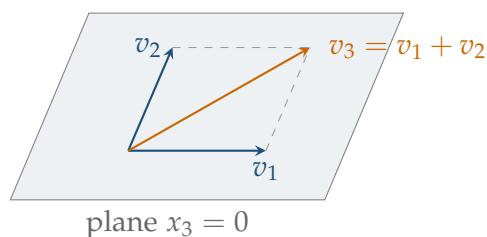
Each pair is independent. By the Note on p. 25, a pair is dependent only if one vector is a multiple of the other. e_1 and e_2 are not multiples of each other (a multiple of $(1, 0, 0)$ has second entry 0, and vice versa), and $(1, 1, 0)$ is not a multiple of $(1, 0, 0)$ or of $(0, 1, 0)$ (it has two nonzero entries; those have one). None of the vectors is 0.

The triple is dependent. $v_1 + v_2 = v_3$, i.e. $1 \cdot v_1 + 1 \cdot v_2 + (-1) \cdot v_3 = 0$, with nonzero weights.

Check with the test: $[v_1 \ v_2 \ v_3] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is already in RREF with pivots in columns 1 and 2 only.

Each pair of columns has 2 pivots, but the triple has only 2 pivots for 3 columns. ✓

Picture: all three vectors lie in the plane $x_3 = 0$ (they are *coplanar*). Two non-parallel vectors in a plane already span it, so the third adds nothing.



The counterexample of p. 30: v_1, v_2, v_3 all lie in one plane through the origin, and v_3 is the diagonal of the parallelogram on v_1, v_2 (parallelogram rule, 8/27 notes, p. 15).

8 Pivotal rows of an RREF (p. 31)

The nonzero rows of an RREF are independent, for the same reason the pivotal columns are: each one has a 1 where the others have 0.

Example (p. 31)

$$R = \begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

has pivots in columns 1, 3, 5. Its pivotal rows (here all three rows) are, as vectors in $\mathbb{R}^5$, $r_1 = (1, 2, 0, -1, 0)$, $r_2 = (0, 0, 1, 3, 0)$, $r_3 = (0, 0, 0, 0, 1)$. Are they independent?

Answer: Yes, the pivotal rows are independent

Suppose $c_1r_1 + c_2r_2 + c_3r_3 = 0$. Entry by entry,

$$c_1r_1 + c_2r_2 + c_3r_3 = (c_1, 2c_1, c_2, -c_1 + 3c_2, c_3) = (0, 0, 0, 0, 0).$$

Look only at the pivot positions 1,3,5: they give $c_1 = 0, c_2 = 0, c_3 = 0$. So only the zero combination gives 0.

Why it works in general: the pivot of row j is a 1, and every other row has 0 in that column (pivots are the only nonzero entries in their columns). So the entry of $c_1r_1 + \cdots + c_r r_r$ in the column of the j -th pivot is exactly c_j .

Check with the test (p. 28): put r_1, r_2, r_3 as the columns of a 5×3 matrix; its RREF is $\begin{bmatrix} I_3 \\ 0 \end{bmatrix}$ (3 pivots in 3 columns), so they are independent. ✓

Summary

Fact	Formula / test
Homogeneous system	$AX = 0$; always has $X = 0$; $A(cX) = cAX$
Nonzero solution exists	$\iff$ infinitely many $\iff$ free variable $\iff$ a column of A has no pivot
$\{v_1, \dots, v_k\}$ independent	$c_1v_1 + \cdots + c_kv_k = 0 \Rightarrow$ all $c_i = 0$
Dependent	some v_i is a combination of the others
Two vectors dependent	$v_1 = cv_2$ or $v_2 = cv_1$
Test	$[v_1 \cdots v_k]$ has a pivot in every column
Always independent	pivotal columns of any matrix; nonzero rows of an RREF; $\{v\}$ with $v \neq 0$
Always dependent	a list containing 0; $k > n$ vectors in $\mathbb{R}^n$
Pitfall	pairwise independent $\nRightarrow$ independent

Corrections to the handwritten notes

Page	As written	Should be	Why
27	$AC = c_{i_1}v_{i_1} + \cdots + c_{i_k}v_k$	$\cdots + c_{i_k}v_{i_k}$	the last pivotal column is v_{i_k} , not v_k