

Linear Algebra Notes — Lecture 9/24

Second-Order Recursions, Complex Numbers & $X_{k+1} = AX_k$

Lecture pages 117–136

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Overview

Last lecture found a formula for the Fibonacci numbers by treating sequences as vectors. This lecture solves *every* recursion of the form

$$x_{k+2} + a_1x_{k+1} + a_0x_k = 0 \quad (a_0 \neq 0)$$

and there are three cases, depending on the roots of $r^2 + a_1r + a_0 = 0$: two real roots, one repeated root, or two complex roots. The third case needs complex numbers, which are reviewed in the middle. The lecture ends by rewriting any such recursion as a matrix equation $X_{k+1} = AX_k$.

Blue boxes answer every blank and question in the handwritten notes; orange boxes are cautions. Typos are listed at the end. (Pages 117–120 repeat the end of the 9/22 lecture; see those notes for

full detail.)

1 Recap: what the Fibonacci example proved (pp. 117–121)

Notation. $\mathcal{S}$ = the vector space of real sequences $(x_k)_{k \geq 0}$, added and scaled term by term.
 $\text{Fib} = \{(x_k) \in \mathcal{S} : x_{k+2} = x_k + x_{k+1} \text{ for all } k\}$.

Theorem (summary of the Fibonacci example)

1. Fib is a subspace of $\mathcal{S}$.
2. The map $T : \text{Fib} \rightarrow \mathbb{R}^2$, $T((x_k)) = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$, is linear and invertible. So $\dim \text{Fib} = 2$.
3. $\{(r_+^k), (r_-^k)\}$ is a basis of Fib, where $r_{\pm} = \frac{1 \pm \sqrt{5}}{2}$ are the roots of $r^2 - r - 1 = 0$.
4. Any $(x_k) \in \text{Fib}$ can be written as $c_+(r_+^k) + c_-(r_-^k)$, where $\begin{bmatrix} c_+ \\ c_- \end{bmatrix}$ are the coordinates of (x_k) in the basis $\{(r_+^k), (r_-^k)\}$.

For the Fibonacci numbers $(x_0 = 0, x_1 = 1)$: $c_{\pm} = \pm \frac{1}{\sqrt{5}}$, so

$$f_k = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^k - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^k.$$

Also, the basis from part 2 is $T^{-1}(e_1) = (1, 0, 1, 1, 2, 3, 5, \dots)$ and $T^{-1}(e_2) = (0, 1, 1, 2, 3, 5, \dots)$.

Homework (p. 120): show $f_k/r_+^k \rightarrow c_+$

Divide by r_+^k : $\frac{f_k}{r_+^k} = c_+ + c_- \left(\frac{r_-}{r_+} \right)^k$. Since $|r_-| \approx 0.618 < 1.618 \approx r_+$, the ratio has absolute value < 1 , so its k -th power tends to 0. Hence $f_k/r_+^k \rightarrow c_+ = 1/\sqrt{5}$. In words, f_k grows like $c_+ r_+^k$.

2 General second-order recursions (p. 121)

Fix real numbers a_1, a_0 with $a_0 \neq 0$, and consider sequences $(x_k) \in \mathcal{S}$ with

$$(*) \quad x_{k+2} + a_1 x_{k+1} + a_0 x_k = 0 \quad \text{for all } k \geq 0.$$

Fibonacci is the case $a_1 = a_0 = -1$ (move everything to one side: $x_{k+2} - x_{k+1} - x_k = 0$).

Theorem

The solutions of $(*)$ form a 2-dimensional subspace H of $\mathcal{S}$.

Proof

Subspace: if $(x_k), (y_k)$ satisfy $(*)$, adding the two equations shows $(x_k + y_k)$ does; multiplying by c shows (cx_k) does; the zero sequence does.

Dimension 2: let $T : H \rightarrow \mathbb{R}^2$, $T((x_k)) = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$. It is linear (it reads off two entries). It is invertible because x_0, x_1 determine everything:

$$x_2 = -a_1 x_1 - a_0 x_0, \quad x_3 = -a_1 x_2 - a_0 x_1, \dots$$

So T is an isomorphism and $\dim H = \dim \mathbb{R}^2 = 2$.

Independence of two geometric sequences (bottom of p. 121)

If $r_+ \neq r_-$ are nonzero, then (r_+^k) and (r_-^k) are independent. Suppose $r_+^k = c r_-^k$ for all k . Then $(r_+/r_-)^k = c$ for all k . At $k = 0$ this says $c = 1$; at $k = 1$ it says $r_+/r_- = 1$, i.e. $r_+ = r_-$, a contradiction.

3 Finding solutions: the characteristic equation (p. 122)

Idea. Search for solutions of the form $(r^k)_{k \geq 0}$ with $r \neq 0$. Plug in:

$$r^{k+2} + a_1 r^{k+1} + a_0 r^k = 0 \xLeftrightarrow{\div r^k} r^2 + a_1 r + a_0 = 0.$$

This quadratic is the **characteristic equation** of $(*)$. By the quadratic formula,

$$r = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}.$$

Why $a_0 \neq 0$ matters here (margin of p. 122)

$r = 0$ is a root exactly when $0^2 + a_1 \cdot 0 + a_0 = 0$, i.e. $a_0 = 0$. (Equivalently, one of the roots $\frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}$ is 0 iff $\sqrt{a_1^2 - 4a_0} = |a_1|$, iff $a_1^2 - 4a_0 = a_1^2$, iff $a_0 = 0$.) So assuming $a_0 \neq 0$ guarantees every root is nonzero, and dividing by r^k is legitimate.

The sign of the **discriminant** $a_1^2 - 4a_0$ splits the problem into three cases:

Case	Discriminant	Roots	Basis of H
1	$a_1^2 > 4a_0$	two distinct real r_+, r_-	$(r_+^k), (r_-^k)$
2	$a_1^2 = 4a_0$	one repeated real $r = -a_1/2$	$(r^k), (k r^k)$
3	$a_1^2 < 4a_0$	two non-real $r_{\pm} = \alpha \pm i\beta$	$(\operatorname{Re} r_+^k), (\operatorname{Im} r_+^k)$

4 Case 1: two distinct real roots (pp. 122–123)

We get two nonzero roots $r_+ \neq r_-$. So $(r_+^k), (r_-^k)$ are two independent vectors (Section 2) in the 2-dimensional space H , hence a basis. Every solution is

$$x_k = c_+ r_+^k + c_- r_-^k,$$

with $c_{\pm}$ found from x_0, x_1 . *Example:* Fibonacci ($a_1 = a_0 = -1$).

Another example. $x_{k+2} - 5x_{k+1} + 6x_k = 0$ with $x_0 = 0, x_1 = 1$. Characteristic equation $r^2 - 5r + 6 = (r-2)(r-3) = 0$, so $r = 2, 3$ and $x_k = c_2 2^k + c_3 3^k$. From $x_0 = 0$: $c_2 + c_3 = 0$. From $x_1 = 1$: $2c_2 + 3c_3 = 1$. So $c_3 = 1, c_2 = -1$:

$$x_k = 3^k - 2^k.$$

Check: $x_2 = 9 - 4 = 5$, and the recursion gives $x_2 = 5x_1 - 6x_0 = 5$. ✓

5 Case 2: one repeated root (pp. 123–124)

If $a_1^2 = 4a_0$ there is only one root, $r = -\frac{a_1}{2}$, and $r \neq 0$ (as $a_0 \neq 0$). The sequence $(r^k) = ((-\frac{a_1}{2})^k)$ solves the recursion, but one vector can't span the 2-dimensional H . We need another.

Try $(x_k) = (k r^k)_{k \geq 0}$.

Plugging in (p. 123, with every step justified)

$$(k+2)r^{k+2} + a_1(k+1)r^{k+1} + a_0 k r^k = r^k \left((k+2)r^2 + a_1(k+1)r + a_0 k \right).$$

Now use the two facts of Case 2:

$$r = -\frac{a_1}{2} \Rightarrow r^2 = \frac{a_1^2}{4}, \quad a_1 r = -\frac{a_1^2}{2} = -2 \cdot \frac{a_1^2}{4}; \quad a_1^2 = 4a_0 \Rightarrow a_0 = \frac{a_1^2}{4}.$$

Substituting,

$$= r^k \left((k+2) \frac{a_1^2}{4} - 2 \cdot \frac{a_1^2}{4} (k+1) + \frac{a_1^2}{4} k \right) = r^k \frac{a_1^2}{4} (k+2 - 2(k+1) + k) = r^k \frac{a_1^2}{4} \cdot 0 = 0. \checkmark$$

Theorem (p. 124)

If $r^2 + a_1 r + a_0 = 0$ (with $a_0 \neq 0$) has a repeated root $r = -\frac{a_1}{2} \neq 0$, then $\{(r^k), (k r^k)\}$ is a basis of the solution space H .

Why it's a basis (not spelled out in the notes)

Both are solutions (above). Apply the isomorphism T (first two terms):

$$T((r^k)) = \begin{bmatrix} 1 \\ r \end{bmatrix}, \quad T((k r^k)) = \begin{bmatrix} 0 \\ r \end{bmatrix}.$$

These are independent in $\mathbb{R}^2$ because $\det \begin{bmatrix} 1 & 0 \\ r & r \end{bmatrix} = r \neq 0$. An isomorphism preserves independence, so the two sequences are independent. Two independent vectors in the 2-dimensional H form a basis.

Example. $x_{k+2} + 2x_{k+1} + x_k = 0$: $r^2 + 2r + 1 = (r + 1)^2 = 0$, so $r = -1$ (repeated). Basis of H : $\{((-1)^k), (k(-1)^k)\}$, and every solution is

$$x_k = c_1(-1)^k + c_2 k(-1)^k = (-1)^k(c_1 + c_2 k).$$

With $x_0 = 1, x_1 = 0$: $c_1 = 1$ and $-(c_1 + c_2) = 0$, so $c_2 = -1$ and $x_k = (-1)^k(1 - k)$, giving $(1, 0, -1, 2, -3, 4, \dots)$. Check: $x_2 = -2x_1 - x_0 = -1$. ✓

6 Review of complex numbers (pp. 125–129)

Case 3 needs roots of equations like $r^2 + 1 = 0$. *Example (p. 125):* $x_{k+2} + x_k = 0$ has characteristic equation $r^2 + 1 = 0$, so $r = \pm i$. (See also the appendix on complex numbers in Lay.)

Definitions

i is a formal solution of $x^2 = -1$: a new symbol with the single rule $i^2 = -1$. A **complex number** is $a + ib$ with a, b real; a is its *real part* and b its *imaginary part*. Notation: $\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}$.

Operations.

1. Addition: $(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$.
2. Multiplication: $(a_1 + ib_1)(a_2 + ib_2) = a_1a_2 + i^2b_1b_2 + i(a_1b_2 + b_1a_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2)$.

Key point. Add and multiply complex numbers as polynomials in the variable i , then replace i^2 by -1 . *Example:* $(1 + 2i)(3 - i) = 3 - i + 6i - 2i^2 = 3 + 5i + 2 = 5 + 5i$.

Definition (complex conjugate)

The complex conjugate of $a + ib$ is $\overline{a + ib} = a - ib$ (flip the sign of the imaginary part).

Example. $(a + ib)(a - ib) = a^2 - (ib)^2 = a^2 - i^2b^2 = a^2 + b^2$. This is a real number, and it is > 0 unless $a = b = 0$.

Inverse of $a + ib$ (the circled part of p. 126)

For $a + ib \neq 0$, multiply top and bottom by the conjugate to make the denominator real:

$$\frac{1}{a + ib} = \frac{a - ib}{(a + ib)(a - ib)} = \frac{a - ib}{a^2 + b^2} = \underbrace{\frac{a}{a^2 + b^2}}_{\text{real part}} + i \underbrace{\frac{-b}{a^2 + b^2}}_{\text{imag. part}}.$$

Example: $\frac{1}{1+2i} = \frac{1-2i}{5}$. Check: $(1+2i) \cdot \frac{1-2i}{5} = \frac{1+4}{5} = 1$. ✓

$a + ib$ is real $\iff b = 0 \iff a + ib = \overline{a + ib}$

$a + ib = a - ib \iff 2ib = 0 \iff b = 0$. So a complex number is real exactly when it equals its own conjugate.

Real and imaginary parts via conjugates (bottom of p. 129). For $z = a + ib$,

$$\frac{z + \bar{z}}{2} = \frac{(a + ib) + (a - ib)}{2} = a = \operatorname{Re} z, \quad \frac{z - \bar{z}}{2i} = \frac{(a + ib) - (a - ib)}{2i} = \frac{2ib}{2i} = b = \operatorname{Im} z.$$

Both are real numbers, for every complex z .

Linear algebra over $\mathbb{C}$ (p. 127)

Everything so far works with complex numbers as scalars (they form a field, p. 107 of the 9/22 notes):

- $\mathbb{C}^n$ = all $n \times 1$ matrices with complex entries;
- vector spaces with complex scalars, e.g. polynomials with complex coefficients, or sequences (x_k) where each x_k is complex.

Theorem (p. 127)

Complex conjugation commutes with addition and multiplication: for $z, w \in \mathbb{C}$,

$$\overline{z + w} = \bar{z} + \bar{w}, \quad \overline{zw} = \bar{z}\bar{w}.$$

Proof (p. 128)

Let $z = a_1 + ib_1$ and $w = a_2 + ib_2$.

Sum: $\overline{z + w} = \overline{(a_1 + a_2) + i(b_1 + b_2)} = (a_1 + a_2) - i(b_1 + b_2) = (a_1 - ib_1) + (a_2 - ib_2) = \bar{z} + \bar{w}$.

Product: $\overline{zw} = \overline{(a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)} = a_1a_2 - b_1b_2 - i(a_1b_2 + a_2b_1)$. On the other side,

$$\bar{z}\bar{w} = (a_1 - ib_1)(a_2 - ib_2) = a_1a_2 + (-i)^2b_1b_2 - i(a_1b_2 + a_2b_1) = a_1a_2 - b_1b_2 - i(a_1b_2 + a_2b_1),$$

using $(-i)^2 = i^2 = -1$. The two agree.

Consequence. Applying the product rule repeatedly gives $\overline{z^k} = \bar{z}^k$ for every $k \geq 0$.

Conjugate of a matrix. $\bar{A}$ means: conjugate every entry. For example

$$\overline{\begin{bmatrix} i & 2+i \\ -1 & -1-i \end{bmatrix}} = \begin{bmatrix} -i & 2-i \\ -1 & -1+i \end{bmatrix}.$$

Corollary (p. 128)

Complex conjugation commutes with addition and multiplication of matrices: $\overline{A+B} = \overline{A} + \overline{B}$ and $\overline{AB} = \overline{A} \overline{B}$.

Proof (p. 129)

Sum: the (i, j) entry of $\overline{A+B}$ is $\overline{a_{ij} + b_{ij}} = \overline{a_{ij}} + \overline{b_{ij}}$ (theorem), which is the (i, j) entry of $\overline{A} + \overline{B}$.

Product: the (i, j) entry of $\overline{AB}$ is row i of $\overline{A}$ times column j of $\overline{B}$:

$$\overline{a_{i1} b_{1j}} + \cdots + \overline{a_{in} b_{nj}} \stackrel{\text{Theorem}}{=} \overline{a_{i1} b_{1j} + \cdots + a_{in} b_{nj}} = (i, j) \text{ entry of } \overline{AB}.$$

7 Case 3: two complex roots (pp. 124, 130–132)

If $a_1^2 < 4a_0$, the discriminant is negative, so $\sqrt{a_1^2 - 4a_0} = i\sqrt{4a_0 - a_1^2}$, and the roots are two distinct non-real complex numbers:

$$r_{\pm} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2} = \frac{-a_1 \pm i\sqrt{4a_0 - a_1^2}}{2}.$$

The same plug-in computation as before (it only used algebra) shows $(r_{\pm}^k)$ and $(r_{\mp}^k)$ solve $x_{k+2} + a_1 x_{k+1} + a_0 x_k = 0$, but as *complex* sequences. We want real solutions.

Example (p. 130). $a_1 = 0$, $a_0 = 1$: $x_{k+2} + x_k = 0$, $r_{\pm} = \pm i$. Indeed $i^{k+2} + i^k = i^k(i^2 + 1) = 0$, so (i^k) and $((-i)^k)$ solve the recursion, where

$$(i^k) = (i^0, i^1, i^2, i^3, i^4, \dots) = (1, i, -1, -i, 1, \dots).$$

Key point (p. 131)

If $(r_{+}^k), (r_{-}^k)$ are non-real solutions of $x_{k+2} + a_1 x_{k+1} + a_0 x_k = 0$, then

$$\left(\frac{r_{+}^k + r_{-}^k}{2} \right) \quad \text{and} \quad \left(\frac{r_{+}^k - r_{-}^k}{2i} \right)$$

are **real** solutions.

Proof (p. 131, with the missing steps)

They are solutions: the recursion is linear, so any combination of solutions with complex coefficients (here $\frac{1}{2}$, $\frac{1}{2}$ and $\frac{1}{2i}$, $-\frac{1}{2i}$) is again a solution.

They are real: from the formula above, $r_{-} = \overline{r_{+}}$ (same real part, opposite imaginary part). By the theorem, $r_{-}^k = \overline{r_{+}^k} = \overline{r_{+}^k}$. Write $r_{+}^k = a + ib$ (for each fixed k). Then

$$\frac{r_{+}^k + r_{-}^k}{2} = \frac{(a + ib) + (a - ib)}{2} = a = \text{Re}(r_{+}^k), \quad \frac{r_{+}^k - r_{-}^k}{2i} = \frac{(a + ib) - (a - ib)}{2i} = b = \text{Im}(r_{+}^k).$$

Both are real.

Why $r_- = \overline{r_+}$ in general: if $r^2 + a_1 r + a_0 = 0$ with a_1, a_0 real, conjugate both sides. Conjugation commutes with $+$ and $\times$ and leaves real numbers alone, so $\bar{r}^2 + a_1 \bar{r} + a_0 = 0$. Non-real roots of real polynomials come in conjugate pairs.

Theorem (p. 131)

$\left(\frac{r_+^k + r_-^k}{2}\right)$ and $\left(\frac{r_+^k - r_-^k}{2i}\right)$ are independent in S .

Proof (pp. 131–132, with the ending made explicit)

Let $u = (r_+^k)$. Then $(r_-^k) = (\overline{r_+^k}) = \bar{u}$, so the two sequences are $\frac{u+\bar{u}}{2}$ and $\frac{u-\bar{u}}{2i}$. Suppose one is a multiple of the other, say $\frac{u+\bar{u}}{2} = c \frac{u-\bar{u}}{2i}$ for a real c . Multiply by $2i$:

$$iu + i\bar{u} = cu - c\bar{u} \implies (c+i)\bar{u} = (c-i)u,$$

that is, $(c+i)r_-^k = (c-i)r_+^k$ for all k . Since c is real, $c \pm i \neq 0$, so

$$\left(\frac{r_+}{r_-}\right)^k = \frac{c+i}{c-i} \text{ is the same constant for every } k.$$

Impossible! At $k = 0$ the left side is 1, so the constant is 1; then $k = 1$ gives $r_+ = r_-$, but the roots are distinct.

The other way round is similar: the first sequence is not a multiple of the second, since its $k = 0$ term is 1 while the second's is 0. So the two sequences are independent.

Quicker check. Apply the isomorphism T (first two terms). Write $r_+ = \alpha + i\beta$ with $\beta \neq 0$. The sequences start $(1, \alpha, \dots)$ and $(0, \beta, \dots)$, and $\det \begin{bmatrix} 1 & 0 \\ \alpha & \beta \end{bmatrix} = \beta \neq 0$.

Theorem (p. 132)

If r_+, r_- are non-real distinct roots of $r^2 + a_1 r + a_0 = 0$, then

$$\left\{ \left(\frac{r_+^k + r_-^k}{2}\right), \left(\frac{r_+^k - r_-^k}{2i}\right) \right\} = \{(\operatorname{Re} r_+^k), (\operatorname{Im} r_+^k)\}$$

is a basis of the (real) solution space of $x_{k+2} + a_1 x_{k+1} + a_0 x_k = 0$.

Why: two independent real solutions in the 2-dimensional space H .

Worked examples

Example 1: $x_{k+2} + x_k = 0$ ($r_{\pm} = \pm i$).

$$\frac{i^k + (-i)^k}{2} : (1, 0, -1, 0, 1, 0, \dots), \quad \frac{i^k - (-i)^k}{2i} : (0, 1, 0, -1, 0, 1, \dots).$$

(E.g. $k = 1$: $\frac{i-(-i)}{2i} = 1$; $k = 3$: $\frac{-i-i}{2i} = -1$.) The first starts $(1, 0)$ and the second $(0, 1)$, so they are exactly $T^{-1}(e_1), T^{-1}(e_2)$. The solution with $x_0 = 2, x_1 = 3$ is therefore $2(1, 0, -1, 0, \dots) + 3(0, 1, 0, -1, \dots) = (2, 3, -2, -3, 2, 3, \dots)$. Check: $x_2 = -x_0 = -2$. ✓

Example 2: $x_{k+2} - 2x_{k+1} + 2x_k = 0$. Here $a_1 = -2, a_0 = 2$, and $a_1^2 = 4 < 8 = 4a_0$, so this is Case 3: $r_{\pm} = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$. Powers of $r_+ = 1 + i$: $1, 1 + i, 2i, -2 + 2i, -4, \dots$ (each is the previous one times $1 + i$; e.g. $(1 + i)^2 = 1 + 2i + i^2 = 2i$). So

$$(\operatorname{Re} r_+^k) = (1, 1, 0, -2, -4, \dots), \quad (\operatorname{Im} r_+^k) = (0, 1, 2, 2, 0, \dots).$$

Check the first with the recursion $x_{k+2} = 2x_{k+1} - 2x_k$: $x_2 = 2 - 2 = 0, x_3 = 0 - 2 = -2, x_4 = -4 - 0 = -4$. ✓

Looking ahead. Case 3 solutions oscillate. Writing $1 + i = \sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$ (polar form, covered later), powers multiply the angle: $r_+^k = 2^{k/2}(\cos \frac{k\pi}{4} + i \sin \frac{k\pi}{4})$. So the basis above is $2^{k/2} \cos \frac{k\pi}{4}$ and $2^{k/2} \sin \frac{k\pi}{4}$: waves whose size grows like $2^{k/2}$.

8 Recursions as matrix equations $X_{k+1} = AX_k$ (pp. 133–134)

Generalize. Let A be an $n \times n$ real matrix, and let $\mathcal{S}^n$ be the vector space of sequences (X_k) of vectors in $\mathbb{R}^n$ (added and scaled term by term). Look at the recursion

$$X_{k+1} = AX_k.$$

Starting from X_1 , its solutions are $(X_1, AX_1, A^2X_1, A^3X_1, \dots)$.

Example: second-order recursions in matrix form. To write $x_{k+2} + a_1x_{k+1} + a_0x_k = 0$ this way, stack two consecutive terms:

$$X_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}, \quad X_{k+1} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}.$$

Why this A: the top entry of X_{k+1} is x_{k+1} , which is the *second* entry of X_k , giving row $(0, 1)$. The bottom entry is $x_{k+2} = -a_0x_k - a_1x_{k+1}$, by solving the recursion for x_{k+2} , giving row $(-a_0, -a_1)$. Check:

$$AX_k = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ -a_0x_k - a_1x_{k+1} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix} = X_{k+1}. \checkmark$$

Theorem (p. 134)

Let A be an $n \times n$ real matrix. The solution space of $X_{k+1} = AX_k$ has dimension n .

Proof, including the “Why?”

Every solution is determined by its first term. Repeatedly substituting,

$$X_{k+1} = AX_k = A(AX_{k-1}) = A^2X_{k-1} = \dots = A^kX_1.$$

The solutions form a subspace of $\mathcal{S}^n$: if $X_{k+1} = AX_k$ and $Y_{k+1} = AY_k$, then $cX_{k+1} + dY_{k+1} = A(cX_k + dY_k)$.

The map. Let T send a solution $(X_k)_{k \geq 1}$ to its first term $X_1 \in \mathbb{R}^n$.

- *Why is T invertible?* Given any $X \in \mathbb{R}^n$, the sequence $(X, AX, A^2X, \dots)$ is a solution with first term X , so T is onto. By the display above, a solution is determined by X_1 , so T is one-to-one. Explicitly, $T^{-1}(X) = (A^{k-1}X)_{k \geq 1}$.
- *T is linear:* $T(c(X_k) + d(Y_k)) = cX_1 + dY_1 = cT((X_k)) + dT((Y_k))$, since operations are term by term.

So T is a linear isomorphism onto $\mathbb{R}^n$, and the solution space has dimension n .

9 Order- n recursions and why $a_0 \neq 0$ (pp. 135–136)

These pages are repeated in the 9/29 notes, where they are worked out in full (including a new example, $x_{k-2} = x_{k-5}$). In brief: the order- n recursion

$$(**) \quad x_{k+n} + a_{n-1}x_{k+n-1} + \dots + a_1x_{k+1} + a_0x_k = 0 \quad (a_0 \neq 0)$$

becomes $X_{k+1} = AX_k$ with $X_k = (x_k, \dots, x_{k+n-1})^T$ and the *companion matrix*

$$A = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & & 0 & 1 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix}.$$

Rows 1 to $n-1$ shift the window up by one; the last row is $(**)$ solved for x_{k+n} . The map $(X_k) \mapsto (x_k)$ (keep the first entry of each X_k) is a linear isomorphism, so by Section 8 the solution space of $(**)$ has dimension n .

Why assume $a_0 \neq 0$? (p. 136)

If $a_0 = 0$ the term x_k disappears, so the recursion is really of lower order than it looks, and the dimension is the true order, not n . That's the professor's boxed example on p. 136:

$$x_{k+4} - x_{k+2} = 0 \quad (n = 4, \text{ but } a_0 = a_1 = 0).$$

Put $j = k + 2$: this says $x_{j+2} = x_j$, an order-2 recursion. Its solutions are the 2-periodic sequences such as $(1, 2, 1, 2, 1, 2, \dots)$, determined by two values, so $\dim = 2$, not 4. Also, with $a_0 \neq 0$:

- every root of the characteristic polynomial is nonzero, so the (r^k) method works (Section 3);
- the companion matrix is invertible ($\det A = \pm a_0$), so the recursion can also be run backwards.

Fine print. If sequences must start at x_0 , the terms x_0, x_1 never appear in $x_{k+4} = x_{k+2}$ (it only involves $x_2, x_3, \dots$), so they would be two extra, meaningless free parameters. Counting them gives 4. The “dim = 2” answer counts only the genuine degrees of freedom of the recursion. Requiring $a_0 \neq 0$ removes this ambiguity.

Example from p. 136: dimension of the solution space of $x_{k+4} - x_{k+2} + x_{k+1} = 0$

The lowest term is x_{k+1} , so $a_0 = 0$. Substituting $j = k + 1$ gives $x_{j+3} - x_{j+1} + x_j = 0$, whose lowest coefficient is $1 \neq 0$: a genuine order-3 recursion. So the dimension is 3.

Note. To actually *solve* $X_{k+1} = AX_k$ we need to compute A^k for all k at once. That is what eigenvalues and eigenvectors are for (next lecture).

Corrections to the handwritten notes

Page	As written	Should be	Why
121	margin: $x_2 = a_1x_1 + a_0x_0$	$x_2 = -a_1x_1 - a_0x_0$	solve (*) for x_2 at $k = 0$
129	scribbles “ $a = cb$ ”, “ $a + ib = (i + 1)a$ ”	(side remarks)	the point is $\operatorname{Re} z = \frac{z+\bar{z}}{2}$, $\operatorname{Im} z = \frac{z-\bar{z}}{2i}$
134	$T : \mathcal{S}^n \rightarrow \mathbb{R}^n$	$T : \{\text{solutions}\} \rightarrow \mathbb{R}^n$	on all of $\mathcal{S}^n$, T isn’t one-to-one
134	sequence $(X_k)_{k \geq 1}$ vs. X_k from $k = 0$ on p. 135	either indexing works	$X_{k+1} = A^k X_1$ if starting at 1; $X_k = A^k X_0$ if at 0
136	$X_{k+1} = AX$	$X_{k+1} = AX_k$	missing index