

Linear Algebra Notes — Lecture 9/29

Recursions, Eigenvectors & Matrices of Linear Maps

Lecture pages 133–148

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Overview

These pages tell one story: a linear recursion can be rewritten as a matrix equation $X_{k+1} = AX_k$, eigenvectors of A give its solutions, and more generally any linear map between finite-dimensional spaces becomes a matrix once bases are chosen. Every blank and open question from the handwritten notes is answered below (in blue boxes), and typos are listed at the end.

The complete 9/29 notes run from p. 133 to p. 145. Pages 146–148 (Sections 11–12) came from an

earlier draft you sent; the 10/1 lecture reuses those page numbers for Markov chains. They are kept here as extra practice.

Notation used throughout

- $\mathbb{R}_{\leq d}[x]$ = polynomials of degree at most d , usual basis $\{1, x, \dots, x^d\}$. Its dimension is $d + 1$ (not d).
- $[v]_\beta$ = coordinate vector of v in the basis β : if $v = c_1 v_1 + \dots + c_n v_n$ then $[v]_\beta = (c_1, \dots, c_n)^T$.
- e_i = i -th standard basis vector of $\mathbb{R}^n$ (1 in slot i , 0 elsewhere).
- For any matrix A : Ae_i = the i -th column of A . This one fact is how every matrix in these notes is found.
- $\text{Nul } A$ = null space (solutions of $Ax = 0$); $\text{rank } A$ = number of pivots = dimension of the column space.
- Rank-nullity: for linear $T : V \rightarrow W$, $\dim \text{Ker } T + \dim \text{Im } T = \dim V$. For an $m \times n$ matrix, $\dim \text{Nul } A + \text{rank } A = n$.

1 Recursions as matrix equations $X_{k+1} = AX_k$ (pp. 133–134)

Generalize. Let A be an $n \times n$ real matrix, and let $\mathcal{S}^n$ be the vector space of sequences (X_k) of vectors in $\mathbb{R}^n$ (added and scaled term by term). Look at the recursion

$$X_{k+1} = AX_k.$$

Starting from X_1 , it forces $X_2 = AX_1$, $X_3 = A^2X_1$, and so on, so every solution looks like

$$(X_1, AX_1, A^2X_1, \dots, A^nX_1, \dots).$$

Example (p. 133): second-order recursions in matrix form. To write $x_{k+2} + a_1x_{k+1} + a_0x_k = 0$ this way, stack two consecutive terms:

$$X_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}, \quad X_{k+1} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}.$$

Why this A: the top entry of X_{k+1} is x_{k+1} , which is the *second* entry of X_k , so the top row is $(0, 1)$ (green in the notes). The bottom entry is $x_{k+2} = -a_0x_k - a_1x_{k+1}$, the recursion solved for x_{k+2} , so the bottom row is $(-a_0, -a_1)$ (red in the notes). Check:

$$AX_k = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ -a_0x_k - a_1x_{k+1} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix} = X_{k+1}. \checkmark$$

Theorem (p. 134)

Let A be an $n \times n$ real matrix. The solution space of $X_{k+1} = AX_k$ has dimension n .

Proof, including the “Why?”

Every solution is determined by its first term. Substituting repeatedly,

$$X_{k+1} = AX_k = A(AX_{k-1}) = A^2X_{k-1} = \cdots = A^kX_1.$$

The solutions form a subspace of $\mathcal{S}^n$: if $X_{k+1} = AX_k$ and $Y_{k+1} = AY_k$, then $c_1X_{k+1} + c_2Y_{k+1} = A(c_1X_k + c_2Y_k)$.

The map. Let T send a solution $(X_k)_{k \geq 1}$ to its first term $X_1 \in \mathbb{R}^n$.

- *Why is T invertible?* The notes’ answer is “ $X_1 \mapsto A^kX_1$.” The inverse takes any starting vector X and builds the sequence $(X, AX, A^2X, \dots)$, which is a solution with first term X . Checking both compositions: starting from X , building the sequence and reading off the first term gives X back. Starting from a solution, reading off X_1 and rebuilding gives the same solution, because $X_{k+1} = A^kX_1$.

- *T is linear (p. 134):* $c_1(X_k) + c_2(Y_k) = (c_1X_k + c_2Y_k) \mapsto c_1X_1 + c_2Y_1 = c_1T((X_k)) + c_2T((Y_k))$.

So T is a linear isomorphism onto $\mathbb{R}^n$, and the solution space has dimension n .

Indexing note. Here the sequence starts at X_1 , so $X_{k+1} = A^kX_1$. Later sections start at X_0 , so $X_k = A^kX_0$. Same idea, shifted by one.

2 Order- n linear recursions (p. 135)

The real sequences (x_k) solving an order- n linear recursion form a vector space of dimension n , because each solution is completely determined by its first n terms.

Setup. Fix real numbers $a_0, a_1, \dots, a_{n-1}$ with $a_0 \neq 0$, and consider sequences satisfying, for every k ,

$$(*) \quad x_{k+n} + a_{n-1}x_{k+n-1} + \cdots + a_1x_{k+1} + a_0x_k = 0.$$

“Order n ” means each term is determined by the n terms before it:

$$x_{k+n} = -a_0x_k - a_1x_{k+1} - \cdots - a_{n-1}x_{k+n-1}.$$

Example: Fibonacci, $x_{k+2} = x_{k+1} + x_k$, is order 2 with $a_1 = -1$, $a_0 = -1$.

Why the solutions form a vector space. If (x_k) and (y_k) satisfy $(*)$, so do $(x_k + y_k)$ and $(c x_k)$, since the left side of $(*)$ is linear in the sequence; the zero sequence is a solution. So the solution set is a subspace of the space of all sequences.

Turning it into a matrix equation. Stack n consecutive terms into a vector (a “window” on the sequence) and slide the window forward by one:

$$X_k = \begin{bmatrix} x_k \\ x_{k+1} \\ \vdots \\ x_{k+n-1} \end{bmatrix}, \quad X_{k+1} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \\ \vdots \\ x_{k+n} \end{bmatrix}.$$

We want A with $X_{k+1} = AX_k$. Build it row by row, asking “what is this entry of X_{k+1} in terms of X_k ?”

- **Rows 1 to $n - 1$:** entry x_{k+i} of X_{k+1} is already the $(i + 1)$ -th entry of X_k , so row i is a single 1 just right of the diagonal (shift the window up by one).
- **Last row:** x_{k+n} is not in X_k , but $(*)$ gives it: $x_{k+n} = -a_0x_k - \cdots - a_{n-1}x_{k+n-1}$. So the last row is $(-a_0, -a_1, \dots, -a_{n-1})$.

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \quad (\text{the companion matrix of } (*)).$$

Check for $n = 2$: $A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}$ and $A \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ -a_0x_k - a_1x_{k+1} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix}$. ✓

Theorem

The map $(X_k) \mapsto (x_k)$ is a linear isomorphism between solutions of $X_{k+1} = AX_k$ and solutions of $(*)$. Hence the solution space of $(*)$ has dimension n .

Why? (p. 135)

1. By Section 1, the solutions of $X_{k+1} = AX_k$ form an n -dimensional space: each is determined by its starting vector, which can be anything in $\mathbb{R}^n$.
2. From (X_k) to (x_k) : take the first entry of each X_k . Back: from (x_k) build the windows X_k . Both maps are linear and undo each other, and the construction of A shows (x_k) solves $(*)$ exactly when (X_k) solves $X_{k+1} = AX_k$.
3. Isomorphisms preserve dimension, so the solution space of $(*)$ has dimension n .

In plain terms: choose $x_0, \dots, x_{n-1}$ freely (n degrees of freedom); $(*)$ forces everything else.

3 Why assume $a_0 \neq 0$? (p. 136)

Answer

The notes' answer: **else we get a “shorter” recursion.** $a_0 \neq 0$ guarantees the recursion genuinely has order n . If $a_0 = 0$ the term x_k drops out of $(*)$, so the equation only links $x_{k+1}, \dots, x_{k+n}$: it is really a recursion of order $n - 1$ (or less) in disguise.

Rule of thumb from p. 136. The *true* order of a recursion is

$$(\text{highest index that appears}) - (\text{lowest index that appears}),$$

and that number is the dimension of the solution space.

Example from p. 136: $x_{k-2} = x_{k-5}$

Highest index $k - 2$, lowest $k - 5$: the order is $(k - 2) - (k - 5) = 3$. Put $m = k - 5$; the recursion says $x_{m+3} = x_m$. So every solution repeats with period 3, like $(1, 2, 3, 1, 2, 3, 1, 2, 3, \dots)$. It is determined by 3 free values (x_0, x_1, x_2) , so the solution space has dimension 3.

Three equivalent ways to see what $a_0 \neq 0$ buys you:

- **The recursion runs backwards.** Solve (*) for x_k : $x_k = -\frac{x_{k+n} + a_{n-1}x_{k+n-1} + \dots + a_1x_{k+1}}{a_0}$, which needs $a_0 \neq 0$. So any n consecutive terms determine the earlier terms as well as the later ones.
- **The companion matrix is invertible.** Expanding along the first column gives $\det A = (-1)^n a_0$. (Check $n = 2$: $\det \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} = 0 \cdot (-a_1) - 1 \cdot (-a_0) = a_0$.) So A is invertible exactly when $a_0 \neq 0$, and then $X_k = A^{-1}X_{k+1}$ steps backwards.
- **Eigenvalues are nonzero.** A is invertible iff 0 is not an eigenvalue, so every eigenvalue satisfies $\lambda \neq 0$ (relevant in Section 5, where solutions look like $\lambda^k v$).

Example: dimension of the solution space of $x_{k+4} - x_{k+2} + x_{k+1} = 0$

Answer: 3

In the standard form (*) with $n = 4$ the coefficients are $a_3 = 0$, $a_2 = -1$, $a_1 = 1$, $a_0 = 0$:

$$x_{k+4} - 0 \cdot x_{k+3} - x_{k+2} + x_{k+1} + 0 \cdot x_k = 0.$$

The lowest term is x_{k+1} , not x_k (the red underlines in the notes), so $a_0 = 0$ and this is *not* a true order-4 recursion. By the rule of thumb, the true order is $(k + 4) - (k + 1) = 3$. Concretely, substitute $j = k + 1$:

$$x_{j+3} - x_{j+1} + x_j = 0.$$

Now the lowest coefficient is $1 \neq 0$: an honest order-3 recursion. Solutions are determined by 3 consecutive values, so the solution space has dimension 3.

Subtlety. If you insist on sequences starting at x_0 with the original indexing, x_0 never appears in any equation, so it is a free, meaningless extra parameter and the count becomes 4. That artificial extra dimension is exactly the degeneracy the assumption $a_0 \neq 0$ rules out.

Note: why we need more linear algebra

Solving $X_{k+1} = AX_k$ means computing $X_k = A^k X_0$ for *all* k at once. Multiplying A by itself k times gives no formula. The way forward is eigenvalues and eigenvectors: on an eigenvector A just scales, so A^k is easy (next section). Later this leads to diagonalization, $A = PDP^{-1}$, which gives $A^k = PD^kP^{-1}$.

4 Eigenvalues and eigenvectors (p. 137)

An eigenvector is a nonzero vector that A only stretches (or flips), never turns; the stretch factor is the eigenvalue.

Definition

Let A be $n \times n$. If $v \in \mathbb{R}^n$, $v \neq 0$, and $Av = \lambda v$ for some number λ , then λ is an **eigenvalue** of A and v is an **eigenvector** of A for λ .

Why $v \neq 0$? $A \cdot 0 = \lambda \cdot 0$ for every λ ; allowing $v = 0$ would make every number an eigenvalue.

Example from the margin of p. 137. If $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $Av = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, then $Av = 3v$, so v is an eigenvector with eigenvalue $\lambda = 3$. To test a vector, check whether Av is a multiple of v ; the multiple is λ .

Another example. $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$: $Ae_1 = 2e_1$ and $Ae_2 = 3e_2$, so e_1 is an eigenvector for $\lambda = 2$ and e_2 for $\lambda = 3$. But $(1, 1)$ is not: $A(1, 1)^T = (2, 3)^T$, not a multiple of $(1, 1)^T$.

Question: is λ real?

Answer: not necessarily

A real matrix can have no real eigenvalues. Take the 90° rotation $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. It turns every nonzero vector by 90° , so no real vector keeps its direction. Algebraically, $A(v_1, v_2)^T = (-v_2, v_1)^T = \lambda(v_1, v_2)^T$ forces $-v_2 = \lambda v_1$ and $v_1 = \lambda v_2$, so $v_1 = -\lambda^2 v_1$. With $v \neq 0$ this gives $\lambda^2 = -1$, i.e. $\lambda = \pm i$ (complex).

This shows up in recursions too: $x_{k+2} + x_k = 0$ has companion matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ with eigenvalues $\pm i$, and its solutions $(1, 0, -1, 0, 1, \dots)$ oscillate rather than grow or shrink. For now the course restricts to real λ ; complex eigenvalues come later.

Example: are nonzero vectors in $\text{Nul } A$ eigenvectors?

Yes, with eigenvalue 0

If $v \neq 0$ and $v \in \text{Nul } A$, then $Av = 0 = 0 \cdot v$. That is exactly $Av = \lambda v$ with $\lambda = 0$. (The notes write “eigenvalue A ”; it should say eigenvalue 0.)

Consequence: 0 is an eigenvalue of $A \iff \text{Nul } A \neq \{0\} \iff A$ is not invertible (Invertible Matrix Theorem, Lay 2.3). This is why an invertible companion matrix ($a_0 \neq 0$) has only nonzero eigenvalues.

Theorem (eigenvectors give solutions)

If $Av = \lambda v$, then $X_k = \lambda^k v$ solves $X_{k+1} = AX_k$.

Proof (“plug in”).

$$AX_k = A(\lambda^k v) = \lambda^k (Av) = \lambda^k (\lambda v) = \lambda^{k+1} v = X_{k+1}.$$

Step 2 uses linearity (scalars pull out of A); step 3 uses $Av = \lambda v$. □

Meaning: starting at an eigenvector the whole future is $X_k = \lambda^k X_0$, no matrix powers needed. If $|\lambda| > 1$ it grows, if $|\lambda| < 1$ it decays, and if $\lambda < 0$ it alternates sign.

5 A basis of eigenvectors gives all solutions (p. 138)

Theorem

Suppose $\{v_1, \dots, v_n\}$ is a basis of $\mathbb{R}^n$ and $Av_i = \lambda_i v_i$, $\lambda_i \neq 0$, for $i = 1, \dots, n$. Then $\{(\lambda_1^k v_1), \dots, (\lambda_n^k v_n)\}$ is a basis of the solution space of $X_{k+1} = AX_k$.

Proof (blanks filled in)

1. **The solution space has dimension** $= n$, by Section 1: a solution is determined by its starting vector, which can be any vector in $\mathbb{R}^n$.
2. **Each** $(\lambda_i^k v_i)$ **is a solution** by the theorem in Section 4.
3. **Are they independent? Yes.** Suppose $c_1(\lambda_1^k v_1) + \dots + c_n(\lambda_n^k v_n)$ is the zero sequence, i.e. equals 0 for every k . At $k = 0$ (where $\lambda^0 = 1$): $c_1 v_1 + \dots + c_n v_n = 0$. The v_i form a basis, hence are independent, so $c_1 = \dots = c_n = 0$.

n independent vectors in an n -dimensional space form a basis. □

About $\lambda_i \neq 0$. The proof never actually uses it. It only avoids the awkward case $\lambda = 0$, where $(\lambda^k v) = (v, 0, 0, \dots)$ depends on the convention $0^0 = 1$. For companion matrices with $a_0 \neq 0$ it holds automatically (previous box).

After this theorem the notes say (p. 139): *we shall discuss eigenvalues/eigenvectors later.*

How to use it. Every solution is

$$X_k = c_1 \lambda_1^k v_1 + c_2 \lambda_2^k v_2 + \dots + c_n \lambda_n^k v_n,$$

and the constants come from the start: at $k = 0$, $X_0 = c_1 v_1 + \dots + c_n v_n$, so $(c_1, \dots, c_n)$ are the coordinates of X_0 in the eigenbasis.

Useful fact for companion matrices. For the companion matrix of $(*)$, $v = (1, \lambda, \lambda^2, \dots, \lambda^{n-1})^T$ is an eigenvector exactly when

$$\lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0 = 0.$$

Reason: Av shifts v up to $(\lambda, \lambda^2, \dots, \lambda^{n-1}, -a_0 - a_1 \lambda - \dots - a_{n-1} \lambda^{n-1})$, which equals λv precisely when the last entry equals λ^n . So the scalar solutions of $(*)$ are $x_k = \lambda^k$ for these roots λ .

Worked example: Fibonacci

Recursion $x_{k+2} - x_{k+1} - x_k = 0$, $x_0 = 0$, $x_1 = 1$ (so $a_0 = -1$, $a_1 = -1$).

Step 1: companion matrix. Last row is $(-a_0, -a_1) = (1, 1)$, so $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

Step 2: eigenvalues. By the fact above, $\lambda^2 - \lambda - 1 = 0$, so

$$\varphi = \frac{1 + \sqrt{5}}{2} \approx 1.618, \quad \psi = \frac{1 - \sqrt{5}}{2} \approx -0.618.$$

Step 3: eigenvectors, checked directly. For $v = (1, \lambda)^T$: $Av = (\lambda, 1 + \lambda)^T = (\lambda, \lambda^2)^T = \lambda(1, \lambda)^T$, using $\lambda^2 = \lambda + 1$. So $v_1 = (1, \varphi)^T$, $v_2 = (1, \psi)^T$, independent since $\varphi \neq \psi$: a basis of $\mathbb{R}^2$.

Step 4: match the start. $X_0 = (0, 1)^T = c_1(1, \varphi)^T + c_2(1, \psi)^T$. First entries: $c_1 + c_2 = 0$. Second: $c_1\varphi + c_2\psi = 1$, so $c_1(\varphi - \psi) = 1$. Since $\varphi - \psi = \sqrt{5}$: $c_1 = 1/\sqrt{5}$, $c_2 = -1/\sqrt{5}$.

Step 5: read off x_k (first entry of X_k):

$$x_k = \frac{\varphi^k - \psi^k}{\sqrt{5}}$$

Check $k = 2$: $\frac{\varphi^2 - \psi^2}{\sqrt{5}} = \frac{(\varphi - \psi)(\varphi + \psi)}{\sqrt{5}} = \frac{\sqrt{5} \cdot 1}{\sqrt{5}} = 1 = x_2$. ✓ Since $|\psi| < 1$, the ψ^k term dies out and $x_k \approx \varphi^k / \sqrt{5}$.

6 The matrix of a linear map (pp. 139–140)

Can we represent T by a matrix? Yes

Any linear $T : V \rightarrow W$ between finite-dimensional spaces is represented by a matrix once a basis of each space is fixed: the i -th column is $[T(v_i)]_\gamma$.

Setup. $T : V \rightarrow W$ linear, $\dim V = n$, $\dim W = m$. Bases $\beta = \{v_1, \dots, v_n\}$ of V and $\gamma = \{w_1, \dots, w_m\}$ of W . The coordinate maps turn abstract vectors into columns of numbers:

- $S : V \rightarrow \mathbb{R}^n$, $S(v) = [v]_\beta$, so $S(c_1v_1 + \dots + c_nv_n) = (c_1, \dots, c_n)^T$;
- $R : W \rightarrow \mathbb{R}^m$, $R(w) = [w]_\gamma$, so $R(d_1w_1 + \dots + d_mw_m) = (d_1, \dots, d_m)^T$.

Both are linear isomorphisms, so $S^{-1} : \mathbb{R}^n \rightarrow V$ exists: $S^{-1}(c_1, \dots, c_n) = c_1v_1 + \dots + c_nv_n$. In particular $S^{-1}(e_i) = v_i$.

The diagram. Instead of going down T directly, go around:

$$\begin{array}{ccc} V & \xrightarrow{S} & \mathbb{R}^n \\ T \downarrow & & \downarrow RTS^{-1} \\ W & \xrightarrow{R} & \mathbb{R}^m \end{array}$$

$RTS^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a composition of linear maps, hence linear, and every linear map $\mathbb{R}^n \rightarrow \mathbb{R}^m$ is multiplication by an $m \times n$ matrix. So $RTS^{-1}(x) = Ax$ for some A .

Which matrix? (p. 140) Use $Ae_i = i$ -th column of A . The notes' quick example: multiplying by e_2 picks out the second column,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 0 + 2 \cdot 1 \\ 3 \cdot 0 + 4 \cdot 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

So to find A , follow each e_i around the diagram:

$$e_i \xrightarrow{S^{-1}} v_i \xrightarrow{T} T(v_i) \xrightarrow{R} [T(v_i)]_\gamma.$$

Theorem

The matrix of T relative to β and γ is the $m \times n$ matrix

$$A = \begin{bmatrix} [T(v_1)]_\gamma & [T(v_2)]_\gamma & \cdots & [T(v_n)]_\gamma \end{bmatrix}, \quad \text{and} \quad [T(v)]_\gamma = A[v]_\beta.$$

Recipe (used in every example below).

1. Apply T to each basis vector v_i of the domain.
2. Write each result as a combination of the codomain basis γ ; the coefficients are $[T(v_i)]_\gamma$.
3. Put these coordinate vectors in as columns, in order.

What it's for. Since S and R are isomorphisms, questions about T become questions about A :

- $\dim \text{Ker } T = \dim \text{Nul } A$, and $v \in \text{Ker } T \iff [v]_\beta \in \text{Nul } A$;
- $\dim \text{Im } T = \text{rank } A$;
- T one-to-one $\iff \text{Nul } A = \{0\} \iff$ every column of A has a pivot;
- T onto $\iff Ax = b$ solvable for every $b \iff$ every row of A has a pivot.

7 Example: differentiation (p. 141)

Setup. $T : \mathbb{R}_{\leq 2}[x] \rightarrow \mathbb{R}_{\leq 2}[x]$, $T(f) = f'$. T is linear because $(f + g)' = f' + g'$ and $(cf)' = cf'$.

Fill-in: the usual basis

$$\beta = \{1, x, x^2\}, \text{ so } \dim \mathbb{R}_{\leq 2}[x] = 3 \text{ and } S(a_0 + a_1x + a_2x^2) = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}.$$

The same basis is used on both sides, so $STS^{-1}(x) = Ax$.

Fill-in: the matrix

Apply the recipe to $1, x, x^2$:

Basis vector	$T(\cdot)$	As a combination of $1, x, x^2$	Column
1	0	$0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$	$(0, 0, 0)$
x	1	$1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$	$(1, 0, 0)$
x^2	$2x$	$0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$	$(0, 2, 0)$

$$A = \begin{bmatrix} [0]_\beta & [1]_\beta & [2x]_\beta \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Check: $A(a_0, a_1, a_2)^T = (a_1, 2a_2, 0)^T$, the coordinates of $a_1 + 2a_2x = f'$. ✓

Q: What is the smallest k with $A^k = 0$?

Answer: $k = 3$

$$A^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq 0, \quad A^3 = A^2 A = 0.$$

For A^2 : row 1 of A is $(0, 1, 0)$, so row 1 of A^2 is row 2 of A , $= (0, 0, 2)$; row 2 of A is $(0, 0, 2)$, so row 2 of A^2 is $2 \cdot (\text{row 3 of } A) = 0$; row 3 is 0. For A^3 : row 1 of A^2 is $(0, 0, 2)$, giving $2 \cdot (\text{row 3 of } A) = 0$.

Why it makes sense. A^k is the matrix of “differentiate k times.” Twice leaves the constant $2a_2$, nonzero for $f = x^2$ (so $A^2 \neq 0$); three times kills every quadratic (so $A^3 = 0$). A matrix with some power equal to 0 is called *nilpotent*.

8 Example: left multiplication $T(B) = CB$ (pp. 142–144)

Setup. C is a fixed $n \times n$ matrix and $T : M_{n \times n}(\mathbb{R}) \rightarrow M_{n \times n}(\mathbb{R})$, $T(B) = CB$.

Is T linear? Yes

By the distributive law: $T(B_1 + B_2) = C(B_1 + B_2) = CB_1 + CB_2 = T(B_1) + T(B_2)$, and $T(cB) = C(cB) = c(CB) = cT(B)$.

How to compute $\dim \text{Ker } T$ and $\dim \text{Im } T$? Represent T by a matrix A (Section 5); then $\dim \text{Ker } T = \dim \text{Nul } A$ and $\dim \text{Im } T = \text{rank } A$.

Sub-example: $n = 2$, $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Fill-in: basis of $M_{2 \times 2}(\mathbb{R})$

The four “matrix units,” each with a single 1:

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Every 2×2 matrix is $\begin{bmatrix} p & q \\ r & s \end{bmatrix} = pE_{11} + qE_{12} + rE_{21} + sE_{22}$, so $\dim M_{2 \times 2} = 4$ and $S\left(\begin{bmatrix} p & q \\ r & s \end{bmatrix}\right) = (p, q, r, s)^T \in \mathbb{R}^4$ (entries read row by row).

Apply T to each basis matrix. Multiplying by E_{ij} on the right copies column i of C into column j :

Basis	$T(\text{basis}) = C \cdot E$	Coordinates (p, q, r, s)
E_{11}	$\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = aE_{11} + cE_{21}$	$(a, 0, c, 0)$
E_{12}	$\begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} = aE_{12} + cE_{22}$	$(0, a, 0, c)$
E_{21}	$\begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix} = bE_{11} + dE_{21}$	$(b, 0, d, 0)$
E_{22}	$\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} = bE_{12} + dE_{22}$	$(0, b, 0, d)$

These are the columns of A (same basis on both sides, $Ax = STS^{-1}(x)$):

$$\begin{array}{ccc} M_{2 \times 2}(\mathbb{R}) & \xrightarrow{S} & \mathbb{R}^4 \\ T \downarrow & & \downarrow Ax=STS^{-1}(x) \\ M_{2 \times 2}(\mathbb{R}) & \xrightarrow{S} & \mathbb{R}^4 \end{array} \quad A = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}.$$

Computing the dimensions. Reorder coordinates as (p, r, q, s) (first column of B , then second). Reordering rows and columns doesn't change rank, and A becomes block diagonal:

$$A \sim \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \implies \text{rank } A = 2 \text{rank } C.$$

The reason: CB acts on each column of B separately, $CB = [Cb_1 \ Cb_2]$. So $B \in \text{Ker } T$ exactly when both columns of B lie in $\text{Nul } C$.

Answer: $\dim \operatorname{Im} T = 2 \operatorname{rank} C$, $\dim \operatorname{Ker} T = 2 \dim \operatorname{Nul} C$

Case	$\operatorname{rank} C$	$\dim \operatorname{Im} T = \operatorname{rank} A$	$\dim \operatorname{Ker} T = \dim \operatorname{Nul} A$
$ad - bc \neq 0$ (C invertible)	2	4	0
$C \neq 0, ad - bc = 0$	1	2	2
$C = 0$	0	0	4

Example: $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has rank 1; $\operatorname{Ker} T =$ matrices with zero first row, spanned by E_{21}, E_{22} : dimension 2. ✓ For general n : $\dim \operatorname{Im} T = n \operatorname{rank} C$ and $\dim \operatorname{Ker} T = n \dim \operatorname{Nul} C$.

Sub-example: if C is invertible, is $T(B) = CB$ invertible? (p. 144)

Answer: yes

$T^{-1}(B) = C^{-1}B$, because $T(C^{-1}B) = CC^{-1}B = B$ and $C^{-1}(CB) = B$. Matrix check: $\operatorname{rank} A = 2 \operatorname{rank} C = 4$, so the 4×4 matrix A is invertible (in fact $\det A = (ad - bc)^2$).

9 Change of basis (pp. 144–145)

Setup. V has two bases $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{u_1, \dots, u_n\}$. This is Section 5 with $T = \operatorname{id}$ (the identity map, $\operatorname{id}(v) = v$) and different bases on the two sides:

$$\begin{array}{ccc} V & \xrightarrow{S} & \mathbb{R}^n \\ \operatorname{id} \downarrow & & \downarrow RS^{-1} \\ V & \xrightarrow{R} & \mathbb{R}^n \end{array} \quad S(v) = [v]_{\beta}, \quad R(v) = [v]_{\gamma}.$$

$RS^{-1}(x) = Ax$, where A is the **change-of-basis matrix**. It converts β -coordinates into γ -coordinates: $[v]_{\gamma} = A[v]_{\beta}$.

Fill-in: $Ae_i = ?$

$Ae_i = RS^{-1}(e_i) = R(v_i) = [v_i]_{\gamma}$. Therefore

$$A = P_{\gamma \leftarrow \beta} = \begin{bmatrix} [v_1]_{\gamma} & [v_2]_{\gamma} & \cdots & [v_n]_{\gamma} \end{bmatrix}.$$

This is Section 4.7 of Lay, which writes it $P_{C \leftarrow B}$. Its inverse converts back: $P_{\beta \leftarrow \gamma} = (P_{\gamma \leftarrow \beta})^{-1}$.

Worked example in $\mathbb{R}^2$ (p. 145)

$\beta = \{v_1, v_2\}$ with $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\gamma = \{w_1, w_2\}$ with $w_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $w_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Here $V = \mathbb{R}^2$, so S and R are just matrices.

S^{-1} . S^{-1} turns coordinates into the vector: $S^{-1}(c_1, c_2) = c_1 v_1 + c_2 v_2$. So $S^{-1}(e_1) = v_1$, $S^{-1}(e_2) = v_2$, and v_1, v_2 are its columns:

$$S^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Fill-in: R^{-1} (and R)

Same reasoning with γ : R^{-1} has columns w_1, w_2 . To get R , use $\begin{bmatrix} p & q \\ r & s \end{bmatrix}^{-1} = \frac{1}{ps - qr} \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$; here $ps - qr = 2 \cdot 0 - 1 \cdot 1 = -1$.

$$R^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \quad R = \frac{1}{-1} \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}.$$

Fill-in: RS^{-1}

$$RS^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 \cdot 1 + 1 \cdot (-1) & 0 \cdot 1 + 1 \cdot 1 \\ 1 \cdot 1 + (-2)(-1) & 1 \cdot 1 + (-2) \cdot 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix}.$$

Check that column i is $[v_i]_\gamma$: column 1 says $v_1 = -w_1 + 3w_2 = (-2 + 3, -1 + 0) = (1, -1) \checkmark$; column 2 says $v_2 = w_1 - w_2 = (2 - 1, 1 - 0) = (1, 1) \checkmark$.

Using it. Take $v = 2v_1 + 3v_2 = (5, 1)$, so $[v]_\beta = (2, 3)$. Then $[v]_\gamma = A(2, 3)^T = (-2 + 3, 6 - 3) = (1, 3)$, and indeed $w_1 + 3w_2 = (2 + 3, 1) = (5, 1)$. $\checkmark$

10 Example: evaluating at two points (pp. 146–147)

Pages 146–148 are not in the complete 9/29 notes, and the 10/1 notes reuse those page numbers for Markov chains. So these two examples came from an earlier draft and may not be covered in lecture. They are kept here as extra practice.

Setup. $T : \mathbb{R}_{\leq 2}[x] \rightarrow \mathbb{R}^2$ with fixed points $a_1 = -1, a_2 = 2$: $T(f) = \begin{bmatrix} f(-1) \\ f(2) \end{bmatrix}$.

Is T linear? Yes

Evaluation respects addition and scaling: $(f + g)(-1) = f(-1) + g(-1)$ and $(cf)(-1) = cf(-1)$, and likewise at 2. So $T(f + g) = T(f) + T(g)$ and $T(cf) = cT(f)$.

Writing T as a matrix. Use $\beta = \{1, x, x^2\}$ with $S(1) = e_1$, $S(x) = e_2$, $S(x^2) = e_3$. The codomain is already $\mathbb{R}^2$ with its standard basis, so no second coordinate map is needed: $TS^{-1}(x) = Ax$.

$$\begin{array}{ccc} \mathbb{R}_{\leq 2}[x] & \xrightarrow{S} & \mathbb{R}^3 \\ \downarrow T & \swarrow TS^{-1}(x)=Ax & \\ \mathbb{R}^2 & & \end{array}$$

Fill-in: the “?” arrows on p. 147, and A

Basis polynomial	value at -1	value at 2	column $T(\cdot)$
1	1	1	$(1, 1)$
x	-1	2	$(-1, 2)$
x^2	1	4	$(1, 4)$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

Check on $f = a_0 + a_1x + a_2x^2$: $A(a_0, a_1, a_2)^T = (a_0 - a_1 + a_2, a_0 + 2a_1 + 4a_2) = (f(-1), f(2))$. ✓

Row reduce ($R_2 \rightarrow R_2 - R_1$, then $R_2 \rightarrow R_2/3$, then $R_1 \rightarrow R_1 + R_2$):

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is T onto? Yes

$T = (x \mapsto Ax) \circ S$ with S a bijection, so T is onto $\iff x \mapsto Ax$ is onto $\iff$ every row of A has a pivot. Both rows have pivots. Meaning: for any targets (y_1, y_2) some polynomial of degree ≤ 2 has $f(-1) = y_1$ and $f(2) = y_2$ (e.g. the line through $(-1, y_1)$ and $(2, y_2)$).

Fill-in: $\dim \text{Ker } T = 1$; is T one-to-one? No

$\dim \text{Ker } T = \dim \text{Nul } A = (\# \text{columns}) - (\# \text{pivots}) = 3 - 2 = 1$. Equivalently, rank-nullity: $3 = \dim \text{Ker } T + 2$. Since $\text{Ker } T \neq \{0\}$, T is *not* one-to-one. (It had to fail: a map from a 3-dimensional space to a 2-dimensional one can't be one-to-one.)

The kernel explicitly. From the reduced matrix, $a_0 = -2a_2$ and $a_1 = -a_2$ with a_2 free. Taking $a_2 = 1$ gives $(-2, -1, 1)$, i.e. $x^2 - x - 2 = (x + 1)(x - 2)$. This makes sense: the kernel is exactly the polynomials vanishing at both -1 and 2 , the multiples of $(x + 1)(x - 2)$.

11 Example: multiplying by a fixed polynomial (p. 148)

Setup. $g = g_0 + g_1x + g_2x^2$ with $g_2 \neq 0$ (degree exactly 2), and $T(f) = gf$.

Fill-in: the codomain $\mathbb{R}_{\leq ?}[x]$

Degrees add when multiplying: $\deg(gf) = \deg g + \deg f \leq 2 + 3 = 5$. So $T : \mathbb{R}_{\leq 3}[x] \rightarrow \mathbb{R}_{\leq 5}[x]$, and $? = 5$.

Is T linear? Yes

$g(f_1 + f_2) = gf_1 + gf_2$ (distributive law) and $g(cf) = c(gf)$.

Is T one-to-one? Yes

Recall: T is one-to-one $\iff \text{Ker } T = \{0\}$. If $gf = 0$ (the zero polynomial) with $f \neq 0$, then gf would have degree $\deg g + \deg f \geq 2$, so it couldn't be zero. Hence $f = 0$ and $\text{Ker } T = \{0\}$.

Why the "Recall" holds: if $T(f_1) = T(f_2)$ then $T(f_1 - f_2) = 0$ by linearity, so $f_1 - f_2 \in \text{Ker } T = \{0\}$, i.e. $f_1 = f_2$.

The matrix version

$\mathbb{R}_{\leq 3}[x]$ has basis $\{1, x, x^2, x^3\}$, so dimension 4; $\mathbb{R}_{\leq 5}[x]$ has basis $\{1, x, \dots, x^5\}$, dimension 6. So $S : \mathbb{R}_{\leq 3}[x] \rightarrow \mathbb{R}^4$, $R : \mathbb{R}_{\leq 5}[x] \rightarrow \mathbb{R}^6$, and $Ax = RTS^{-1}(x)$ with A of size 6×4 :

$$\begin{array}{ccc} \mathbb{R}_{\leq 3}[x] & \xrightarrow{S} & \mathbb{R}^4 \\ \downarrow T & & \downarrow Ax=RTS^{-1}(x) \\ \mathbb{R}_{\leq 5}[x] & \xrightarrow{R} & \mathbb{R}^6 \end{array}$$

Columns: multiplying g by x^j shifts its coefficients down j places.

Basis	$T(\text{basis}) = g x^j$	Column (coefficients of $1, x, \dots, x^5$)
1	$g_0 + g_1x + g_2x^2$	$(g_0, g_1, g_2, 0, 0, 0)$
x	$g_0x + g_1x^2 + g_2x^3$	$(0, g_0, g_1, g_2, 0, 0)$
x^2	$g_0x^2 + g_1x^3 + g_2x^4$	$(0, 0, g_0, g_1, g_2, 0)$
x^3	$g_0x^3 + g_1x^4 + g_2x^5$	$(0, 0, 0, g_0, g_1, g_2)$

$$A = \begin{bmatrix} g_0 & 0 & 0 & 0 \\ g_1 & g_0 & 0 & 0 \\ g_2 & g_1 & g_0 & 0 \\ 0 & g_2 & g_1 & g_0 \\ 0 & 0 & g_2 & g_1 \\ 0 & 0 & 0 & g_2 \end{bmatrix}$$

Fill-in: $\text{Nul } A = \{0\}$

The bottom four rows (rows 3–6) form an upper-triangular 4×4 block with $g_2 \neq 0$ on every diagonal entry, so every column has a pivot. Concretely, $Ax = 0$ forces $x_4 = 0$ from row 6, then $x_3 = 0$ from row 5, and so on up.

Consequences: $\dim \text{Ker } T = 0$ and $\dim \text{Im } T = \text{rank } A = 4$. T is *not onto*: its image is 4-dimensional inside the 6-dimensional $\mathbb{R}_{\leq 5}[x]$ (e.g. the constant 1 is not a multiple of g , since gf has degree ≥ 2 whenever $f \neq 0$).

The handwriting shows $\mathbb{R}^3$, $\mathbb{R}^5$ and a 5×3 matrix. Those numbers would fit if f had degree ≤ 2 (3 coefficients) and gf degree ≤ 4 (5 coefficients). The argument and conclusion (one-to-one, $\text{Nul } A = \{0\}$) are identical either way.

Corrections to the handwritten notes

Small slips in the original pages; these notes use the corrected versions.

Page	As written	Should be	Why
135	$a_0 \neq 0$ squeezed in above the equation	part of the hypothesis	needed for a true order- n recursion (Section 2)
134	$T : S^n \rightarrow \mathbb{R}^n$	$T : \{\text{solutions}\} \rightarrow \mathbb{R}^n$	on all of S^n , T isn't one-to-one
137	nonzero $v \in \text{Nul } A$: "eigenvalue A "	eigenvalue 0	$Av = 0 = 0 \cdot v$
138	$Av_i = \lambda_i v_i, i = 1, \dots, k$	$i = 1, \dots, n$	there are n basis vectors; k is the time index
148	$\mathbb{R}^3, \mathbb{R}^5, A$ is 5×3	$\mathbb{R}^4, \mathbb{R}^6, 6 \times 4$	$\mathbb{R}_{\leq d}[x]$ has dimension $d + 1$