

Linear Algebra Notes — Lecture 9/3

Linear Transformations, the Standard Matrix, Onto and One-to-One

Lecture pages 32–46 · Lay §1.8–1.9

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Overview

Until now a matrix A was a tool for writing a linear system $Ax = b$. This lecture turns the point of view around: A becomes a *function* $x \mapsto Ax$ that takes vectors in $\mathbb{R}^n$ to vectors in $\mathbb{R}^m$. The facts about pivots from 8/27 and 9/1 then turn into facts about this function. The steps:

1. Recall domain, codomain and image of a function (pp. 32–33).

2. The main example $T(x) = Ax$; its image is the span of the columns (pp. 33–34), plus a gallery of examples: zero map, identity, scaling, projections, reflections (pp. 34–36).
3. The definition of a *linear* map, and the fact that $x \mapsto Ax$ is linear (pp. 37–39).
4. Every linear map $\mathbb{R}^n \rightarrow \mathbb{R}^m$ is a matrix map $x \mapsto Ax$; the matrix is the standard matrix $A = [T(e_1) \cdots T(e_n)]$ (pp. 40–42). Example: rotations (pp. 43–44).
5. Onto and one-to-one, and how to read them off from the pivots of A (pp. 44–46).

Every blank and question in the notes is answered in a blue box. Typos are listed at the end.

Notation used throughout.

- $f : X \rightarrow Y$: a function with domain X and codomain Y .
- a_i (or v_i): column i of the matrix A , so $A = [a_1 \cdots a_n]$.
- $e_1, \dots, e_n$: the standard basis vectors of $\mathbb{R}^n$ (columns of the identity I_n), p. 40.
- $\mathbb{R} - \{0\}$: all real numbers except 0.

Recipe: the standard matrix of a linear map $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$

Compute $T(e_1), \dots, T(e_n)$ and use them as the columns: $A = [T(e_1) \cdots T(e_n)]$, an $m \times n$ matrix. Then $T(x) = Ax$ for every x . Then: T is onto $\iff$ every row of A is pivotal; T is one-to-one $\iff$ every column of A is pivotal.

1 Functions: domain, codomain, image (pp. 32–33)

A function has three pieces of data: where inputs live, where outputs live, and which outputs actually occur.

The lecture covers Lay §1.8 and maybe §1.9 (p. 32). Recall from precalculus: a *function* f is a device with input x and output $f(x)$.

Definition (domain, codomain, image; pp. 32)

- The **domain** of f is the space of inputs.
- The **codomain** of f is the space where the outputs live.
- The **image** (also called the **range**) of f is the set $\{f(x) : x \text{ is in the domain}\}$ of outputs that actually occur.

Today the functions are $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$. For example a function of two variables $f(x, y)$ with real values is a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$.

The image is always inside the codomain, but it can be smaller. The codomain is a choice made when the function is defined. The image is computed from the formula.

Domain and codomain of $f(x) = x^2$: both are $\mathbb{R}$ (p. 32)

The notes draw the parabola $y = x^2$ with the horizontal axis labelled “domain” and the vertical axis labelled “codomain”. Any real x can be squared, so the domain is $\mathbb{R}$. The output is a real number, so the codomain is $\mathbb{R}$. In symbols $f : \mathbb{R} \rightarrow \mathbb{R}$.

Image of $f(x) = x^2$: $[0, \infty)$, all $y \geq 0$ (p. 32)

A square is never negative, so every output is ≥ 0 . Conversely every $y \geq 0$ occurs: $y = f(\sqrt{y})$. So the image is $\{y : y \geq 0\} = [0, \infty)$, which is smaller than the codomain $\mathbb{R}$. Check: $f(\sqrt{5}) = 5$; and -1 is not an output, since $x^2 = -1$ has no real solution. ✓

Domain, codomain, image of $f(x) = 1/x$: $\mathbb{R} - \{0\}$, $\mathbb{R}$, $\mathbb{R} - \{0\}$ (p. 33)

The notes give the answer: domain = $\mathbb{R} - \{0\}$, codomain = $\mathbb{R}$, image = $\mathbb{R} - \{0\}$. The reasons:

- *Domain*: $1/x$ makes sense exactly when $x \neq 0$.
- *Codomain*: the outputs are real numbers.
- *Image*: $1/x$ is never 0 (a fraction with numerator 1 cannot vanish). Every $y \neq 0$ occurs, since $f(1/y) = y$.

Check: $f(1/3) = 3$ and $f(-1/2) = -2$; the graph (a hyperbola) never touches the x -axis. ✓

2 The main example: $T(x) = Ax$ and its image (pp. 33–34)

Every $m \times n$ matrix A gives a function $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, and its image is the span of the columns of A .

Main example (p. 33). Let A be an $m \times n$ matrix. Define $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ by $T(x) = Ax$. The product Ax needs x to have n entries (one per column of A) and gives a vector with m entries (one per row). So

$$\text{domain of } T = \mathbb{R}^n, \quad \text{codomain of } T = \mathbb{R}^m.$$

Theorem (p. 33)

If $A = [a_1 \cdots a_n]$, then the image of $T(x) = Ax$ is $\text{span}\{a_1, \dots, a_n\}$.

Proof (p. 34, written out)

Here a_i is column i of A . By the definition of the matrix–vector product (8/27 notes, p. 18),

$$T(x) = [a_1 \cdots a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 a_1 + \cdots + x_n a_n.$$

So every output is a linear combination of $a_1, \dots, a_n$ with weights $x_1, \dots, x_n$, and the image

is inside the span. Conversely, any linear combination $c_1a_1 + \cdots + c_na_n$ equals $T(c)$ for $c = (c_1, \dots, c_n)$. Since any x is allowed, the outputs cover *all* linear combinations of $a_1, \dots, a_n$. This is the same statement as the Note on p. 18: $\text{span}\{a_1, \dots, a_n\} = \{Ax : x \in \mathbb{R}^n\}$.

3 A gallery of matrix maps (pp. 34–36)

Simple matrices give simple geometric maps: kill everything, do nothing, stretch, project, reflect.

Zero matrix (p. 34)

If A is the zero matrix, then $T(x) = Ax = 0$ for all x . Its image is just $\{0\}$ (the span of zero columns).

Identity matrix (p. 34)

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with A the $n \times n$ **identity matrix** I_n : ones on the diagonal, zeros elsewhere. Column i of I_n has a 1 in entry i and zeros elsewhere, so

$$T(x) = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x.$$

So $T(x) = x$: the identity matrix gives the map that does nothing. (The notes mark the sum “= x ”.)

Scalar matrix (pp. 34–35)

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with

$$A = \begin{bmatrix} c & & 0 \\ & \ddots & \\ 0 & & c \end{bmatrix} \quad (c \text{ on the diagonal, zeros elsewhere})$$

for some scalar c . The same computation gives $T(x) = x_1(ce_1) + \cdots + x_n(ce_n) = cx$. So T is **scaling by c** : it stretches every vector by the factor c (and flips it if $c < 0$).

Projections (p. 35)

Projection of $\mathbb{R}^2$ onto the x_1 -axis. It keeps the first coordinate and sets the second to 0:

$$T \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.$$

The first row $(1, 0)$ copies x_1 ; the second row $(0, 0)$ outputs 0.

Projection of $\mathbb{R}^2$ onto the x_2 -axis. $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$.

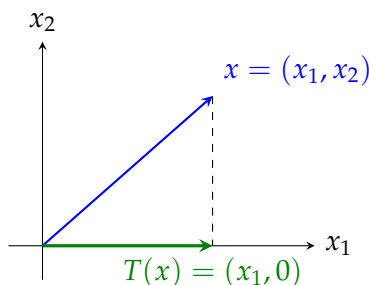


Figure (p. 35): the projection onto the x_1 -axis drops x straight down onto the horizontal axis.

Reflections (p. 36)

Reflection in which line? $(x_1, x_2) \mapsto (x_1, -x_2)$: the x_1 -axis; $(x_1, x_2) \mapsto (x_2, x_1)$: the line $x_2 = x_1$

The notes answer both on p. 36.

- $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$. The first coordinate stays and the second changes sign, so points above the x_1 -axis go to the mirror point below it. The line of the mirror is the x_1 -axis; its points $(x_1, 0)$ do not move.
- $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$. Swapping the coordinates is the mirror in the diagonal line $\{x_2 = x_1\}$; its points (t, t) do not move.

Check: $(3, 1) \mapsto (3, -1)$ for the first map and $(3, 1) \mapsto (1, 3)$ for the second; in both cases the midpoint $((3, 0)$, resp. $(2, 2))$ lies on the mirror line. ✓

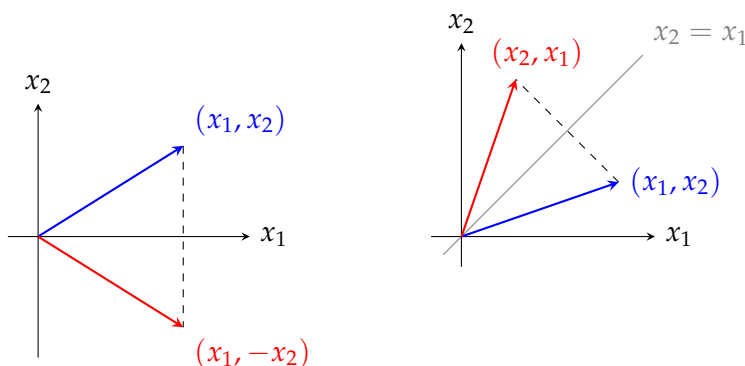


Figure (p. 36): left, reflection in the x_1 -axis; right, reflection in the line $x_2 = x_1$. The dashed segment joins a vector to its mirror image and is cut in half by the mirror line.

4 Linear maps (pp. 37–39)

A map is linear when it respects sums and scalar multiples; matrix maps $x \mapsto Ax$ are linear, and linear maps send linear combinations to linear combinations and 0 to 0.

Definition (linear map; p. 37, the key concept)

A function $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ (also called a **map** or **transformation**) is **linear** if

- (a) $T(x + y) = T(x) + T(y)$ for all x, y in $\mathbb{R}^n$;
- (b) $T(cx) = cT(x)$ for all x in $\mathbb{R}^n$ and c in $\mathbb{R}$.

Example (p. 37): $T(x) = Ax$ is linear

(a) Write $A = [a_1 \ \cdots \ a_n]$. The vector $x + y$ has entries $x_i + y_i$, so

$$\begin{aligned} T(x + y) &= A(x + y) = [a_1 \ \cdots \ a_n] \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} = (x_1 + y_1)a_1 + \cdots + (x_n + y_n)a_n \\ &= x_1a_1 + \cdots + x_na_n + y_1a_1 + \cdots + y_na_n = Ax + Ay. \end{aligned}$$

The first line is the definition of Ax (8/27 notes, p. 18); the second line distributes each $(x_i + y_i)a_i$ and regroups.

(b) $T(cx) = A(cx) = cAx = cT(x)$. The middle step is the “key property” from the 9/1 notes, p. 22: $A(cx) = cx_1a_1 + \cdots + cx_na_n = c(x_1a_1 + \cdots + x_na_n)$.

Check with $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $x = (1, 0)$, $y = (0, 1)$: $A(x + y) = A(1, 1)^T = (3, 7)^T$ and $Ax + Ay = (1, 3)^T + (2, 4)^T = (3, 7)^T$. ✓

Theorem (p. 38)

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a map. Then T is linear if and only if

$$T(c_1v_1 + \cdots + c_kv_k) = c_1T(v_1) + \cdots + c_kT(v_k)$$

for all vectors $v_1, \dots, v_k$ (and every k) and all scalars $c_1, \dots, c_k$.

In words: T is linear iff T takes linear combinations to linear combinations (with the same weights).

Proof (p. 38, both directions written out)

($\Leftarrow$) Assume the displayed formula holds for every k . Take $k = 2$ and $c_1 = 1 = c_2$: it says $T(v_1 + v_2) = T(v_1) + T(v_2)$, which is (a). Take $k = 1$: it says $T(c_1v_1) = c_1T(v_1)$, which is (b). So T is linear.

($\Rightarrow$) Assume T is linear; induct on k . For $k = 1$ the formula is (b). Suppose the formula holds for $k - 1$ vectors (induction hypothesis). Then

$$\begin{aligned} T(c_1v_1 + \cdots + c_kv_k) &\stackrel{(a)}{=} T(c_1v_1 + \cdots + c_{k-1}v_{k-1}) + T(c_kv_k) \\ &= c_1T(v_1) + \cdots + c_{k-1}T(v_{k-1}) + T(c_kv_k) \quad (\text{induction hypothesis}) \\ &\stackrel{(b)}{=} c_1T(v_1) + \cdots + c_{k-1}T(v_{k-1}) + c_kT(v_k). \end{aligned}$$

Step 1 applies (a) with $x = c_1v_1 + \cdots + c_{k-1}v_{k-1}$ and $y = c_kv_k$.

Example (p. 39). Suppose T is linear and $T(v_1) = 0 = T(v_2)$. Then T takes all of $\text{span}\{v_1, v_2\}$ to 0:

$$T(c_1v_1 + c_2v_2) \stackrel{\text{linear}}{=} c_1 \underbrace{T(v_1)}_0 + c_2 \underbrace{T(v_2)}_0 = 0.$$

Theorem (p. 39)

Any linear map takes the zero vector to the zero vector: $T(0) = 0$.

Proof (p. 39)

The notes give two arguments.

- By (a): $T(y) = T(0 + y) = T(0) + T(y)$. Subtract $T(y)$ from both sides: $T(0) = 0$.
- By (b) with $c = 0$: $T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0$.

(In the notes the first argument is partly crossed out, and the second one is the one kept.)

A quick test for non-linearity. If $T(0) \neq 0$, then T is *not* linear. The converse fails: $T(x) = x^2$ on $\mathbb{R}$ has $T(0) = 0$ but $T(2) = 4 \neq 2T(1)$, so (b) fails.

5 Is b in the image? (pp. 39–40)

Asking whether b is in the image of $x \mapsto Ax$ is the same as asking whether $Ax = b$ is consistent.

Example (p. 39). Let $T(x) = Ax$ where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 7 \\ 5 \\ 7 \end{bmatrix}.$$

A has 2 columns, so the domain of T is $\mathbb{R}^2$. A has 3 rows, so the codomain is $\mathbb{R}^3$.

Is b in the image of T ? Yes. The only x with $T(x) = b$ is $x = (2, 5)$ (pp. 39–40)

b is in the image exactly when $Ax = T(x) = b$ has a solution. Row reduce the augmented

matrix as in the notes (subtract row 1 from row 3, then row 2 from row 1):

$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 5 \\ 1 & 1 & 7 \end{array} \right] \xrightarrow{R_3 - R_1} \left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array} \right] \xrightarrow{R_1 - R_2} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array} \right].$$

There is no pivot in the last column, so the system is consistent (8/27 notes, p. 10): **yes**, b is in the image. Both columns are pivotal, so there are no free variables and the solution is unique (8/27 notes, p. 10): $x_1 = 2, x_2 = 5$. So the set of all x taken by T to b is $\{(2, 5)\}$.

Check: $A \begin{bmatrix} 2 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 7 \end{bmatrix} = b. \checkmark$

Meaning: $b = 2a_1 + 5a_2$ lies in the plane $\text{span}\{a_1, a_2\}$, the image of T (Theorem p. 33).

6 Standard basis and the standard matrix (pp. 40–42)

A linear map is completely determined by what it does to $e_1, \dots, e_n$, and those outputs are the columns of its matrix.

Definition (standard basis vectors; p. 40)

The **standard basis vectors** $e_1, \dots, e_n$ in $\mathbb{R}^n$ are the columns of the $n \times n$ identity matrix I_n . Namely $e_i = \text{column } i \text{ of } I_n$: a 1 in entry i , zeros elsewhere.

For $n = 2$: $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. For $n = 3$: $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Key property (p. 41). Every vector is a combination of the e_i , with its own entries as weights:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 e_1 + \dots + x_n e_n \quad \text{or, the same thing,} \quad = I_n \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

(This is the identity-matrix computation from p. 34.)

Another cool property (p. 41). If $A = [v_1 \ \dots \ v_n]$ is an $m \times n$ matrix with $v_i = \text{column } i \text{ of } A$, then

$$Ae_i = v_i, \quad \text{i.e. for } T(x) = Ax: \quad T(e_i) = v_i.$$

For example $[v_1 \ \dots \ v_n] \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 1v_1 + 0v_2 + \dots + 0v_n = v_1$. Multiplying by e_i picks out column i .

Theorem (standard matrix of a linear map; p. 41, Lay §1.9)

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map and let

$$A = [T(e_1) \ \dots \ T(e_n)],$$

the $m \times n$ matrix whose i -th column is $T(e_i)$. Then $T(x) = Ax$ for all x in $\mathbb{R}^n$. A is called the

standard matrix of T .

Proof (p. 42, steps labelled)

$$T(x) \stackrel{(1)}{=} T(x_1 e_1 + \cdots + x_n e_n) \stackrel{(2)}{=} x_1 T(e_1) + \cdots + x_n T(e_n) \stackrel{(3)}{=} [T(e_1) \cdots T(e_n)] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = Ax.$$

(1) is the key property of $e_1, \dots, e_n$ (p. 41). (2) is linearity, in the form of the Theorem on p. 38. (3) is the definition of a matrix times a vector, read backwards (8/27 notes, p. 18).

Conclusion (p. 42)

Linear maps $\mathbb{R}^n \rightarrow \mathbb{R}^m$ are another description of $m \times n$ matrices: every matrix A gives the linear map $x \mapsto Ax$ (p. 37), and every linear map is of this form for exactly one A , namely $A = [T(e_1) \cdots T(e_n)]$ (p. 41).

Why “exactly one”: if $T(x) = Bx$ for all x , then column i of B is $Be_i = T(e_i)$, so $B = A$.

7 How to check if a map is linear (p. 42)

To test a map $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, check whether $T(x) = Ax$ for some matrix A ; the only candidate is $A = [T(e_1) \cdots T(e_n)]$.

Is $T(x_1, x_2) = (x_1 + x_2, x_2 + 1)$ linear? No

The notes' method. If T were linear, its matrix would have to be

$$\left[T \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \quad T \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \right] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},$$

since $T(1, 0) = (1 + 0, 0 + 1) = (1, 1)$ and $T(0, 1) = (0 + 1, 1 + 1) = (1, 2)$. But

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix} \neq \begin{bmatrix} x_1 + x_2 \\ x_2 + 1 \end{bmatrix}$$

(the notes mark this with an $\times$; e.g. at $x = 0$ the second entries are 0 and 1). By the Theorem on p. 41 a linear T would equal this matrix map, so T is not linear.

Faster: $T(0, 0) = (0, 1) \neq (0, 0)$, so T is not linear by the Theorem on p. 39.

Check that (a) fails: $T(1, 0) + T(0, 1) = (1, 1) + (1, 2) = (2, 3)$, but $T(1, 1) = (2, 2)$. $\checkmark$

The notes add: a lot of times linearity of T is easy to check directly from the definition (a), (b).

8 Rotations (pp. 43–44)

Rotation by an angle φ is linear, and its standard matrix is $\begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix}$.

Example (p. 43). Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ rotate the vector x counterclockwise by the angle φ , with φ in $(0, \pi/2)$.

Why T is linear (the two pictures on p. 43). (b): stretching x to cx and then rotating gives the same arrow as rotating first and then stretching, so $T(cx) = cT(x)$. (a): $x + y$ is the diagonal of the parallelogram spanned by x and y (parallelogram rule, 8/27 notes, p. 15). Rotating moves the whole parallelogram rigidly, so the rotated diagonal is the diagonal of the rotated parallelogram: $T(x + y) = T(x) + T(y)$.

Standard matrix of T : columns $T(e_1) = (\cos \varphi, \sin \varphi)$ and $T(e_2) = (-\sin \varphi, \cos \varphi)$ (pp. 43–44)

By the recipe, we need $T(e_1)$ and $T(e_2)$. Both e_1 and e_2 have length 1, and so do their rotations.

- $T(e_1)$: rotating $e_1 = (1, 0)$ by φ gives the arrow of length 1 at angle φ above the x_1 -axis. In the right triangle with hypotenuse 1 and angle φ , the adjacent side is $\cos \varphi$ and the opposite side is $\sin \varphi$, so $T(e_1) = \begin{bmatrix} \cos \varphi \\ \sin \varphi \end{bmatrix}$.
- $T(e_2)$: rotating $e_2 = (0, 1)$ by φ tilts it to the left of the x_2 -axis. The same triangle, now standing on the x_2 -axis, has horizontal side $\sin \varphi$ (pointing left, hence negative) and vertical side $\cos \varphi$, so $T(e_2) = \begin{bmatrix} -\sin \varphi \\ \cos \varphi \end{bmatrix}$.

Hence

$$[T(e_1) \ T(e_2)] = \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix}.$$

Check with $\varphi = \pi/2$ (a quarter turn): the matrix is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, which sends $e_1 \mapsto e_2$ and $e_2 \mapsto -e_1$, exactly a quarter turn counterclockwise. Also the image $(\cos \varphi - \sin \varphi, \sin \varphi + \cos \varphi)$ of $(1, 1)$ has length $\sqrt{2}$, like $(1, 1)$ itself. ✓

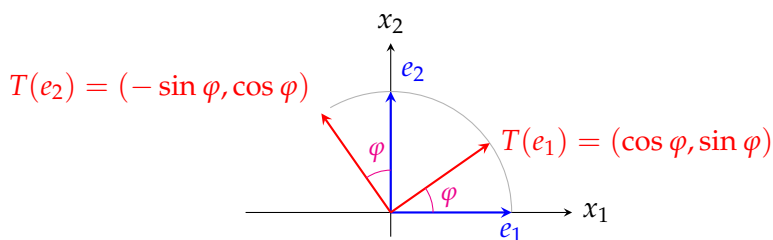


Figure (p. 43): rotating e_1 and e_2 by φ ; the red arrows are the columns of the standard matrix.

The notes add (p. 44): one can show the matrix works for any φ , not just φ in $(0, \pi/2)$. (Our sketch of why:) by the definition of cosine and sine on the unit circle, rotating e_1 by any angle φ lands at $(\cos \varphi, \sin \varphi)$. And e_2 is e_1 rotated by $\pi/2$, so $T(e_2) = (\cos(\varphi + \frac{\pi}{2}), \sin(\varphi + \frac{\pi}{2})) = (-\sin \varphi, \cos \varphi)$ by the angle-addition formulas from precalculus.

Replace φ by $-\varphi$: the off-diagonal signs swap; this is rotation clockwise by φ , which undoes T

Use $\cos(-\varphi) = \cos \varphi$ and $\sin(-\varphi) = -\sin \varphi$ (cosine is even, sine is odd):

$$\begin{bmatrix} \cos(-\varphi) & -\sin(-\varphi) \\ \sin(-\varphi) & \cos(-\varphi) \end{bmatrix} = \begin{bmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{bmatrix}.$$

Rotating counterclockwise by $-\varphi$ means rotating *clockwise* by φ . This undoes T : rotate by φ , then back by φ , and you are where you started. In the matrix the off-diagonal signs swap.

Check: apply the new matrix to $T(e_1) = (\cos \varphi, \sin \varphi)$: $(\cos^2 \varphi + \sin^2 \varphi, -\sin \varphi \cos \varphi + \cos \varphi \sin \varphi) = (1, 0) = e_1$. ✓

9 Onto maps (pp. 44–45)

A map is onto when every vector of the codomain is hit; for $x \mapsto Ax$ this happens exactly when every row of A has a pivot.

Definition (onto; p. 44)

A function f is **onto** if image of f = codomain of f . Thus $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is onto if for every b in $\mathbb{R}^m$ there is an x in $\mathbb{R}^n$ with $T(x) = b$.

Example (p. 44): $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3$ is onto, since any b equals $f(\sqrt[3]{b})$. By contrast x^2 is not onto (its image is $[0, \infty)$, p. 32).

Theorem (p. 45, Lay §1.9)

If $T(x) = Ax$, then T is onto $\iff$ each row of A is pivotal.

Proof (p. 45)

T is onto exactly when $Ax = b$ is consistent for every b in $\mathbb{R}^m$. By the Theorem on p. 19 (8/27 notes, pp. 19–20), $Ax = b$ is consistent for every $b \iff A$ has a pivot in every row.

Is rotation of $\mathbb{R}^2$ onto? Yes

Pivots. Write $c = \cos \varphi$, $s = \sin \varphi$, so $A = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$.

- If $c \neq 0$: $R_2 - \frac{s}{c}R_1$ gives the second row $(0, c + \frac{s^2}{c}) = (0, \frac{1}{c})$, using $c^2 + s^2 = 1$. Both rows have pivots.
- If $c = 0$: then $s = \pm 1$ and $A = \begin{bmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{bmatrix}$; swapping the rows shows a pivot in each row.

So every row is pivotal and T is onto by the Theorem on p. 45.

Geometrically: given b , rotate it back by φ (the matrix with $-\varphi$ from p. 44) to get x ; then $T(x) = b$. *Check* with $\varphi = \pi/2$, $b = (3, 1)$: $x = (1, -3)$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} (1, -3)^T = (3, 1)^T$. ✓

Is projection of $\mathbb{R}^2$ onto a coordinate axis onto? No

For the projection onto the x_1 -axis, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is already in RREF and its second row has no pivot, so T is not onto. Concretely, $b = (0, 1)$ is not an output: $T(x_1, x_2) = (x_1, 0)$ always has second entry 0. The image is only the x_1 -axis, not all of the codomain $\mathbb{R}^2$. The projection onto the x_2 -axis is the same with the roles of the rows swapped ($b = (1, 0)$ is missed).

Two meanings of “onto”. “Projection *onto* the x_1 -axis” is everyday language: the outputs land on that axis. “ T is onto” is the technical word of the definition above: the outputs fill the whole codomain. The projection lands onto the axis but is not onto $\mathbb{R}^2$.

10 One-to-one maps (pp. 45–46)

A map is one-to-one when different inputs give different outputs; for $x \mapsto Ax$ this happens exactly when every column of A has a pivot.

Definition (one-to-one; p. 45)

A function f is **one-to-one** if $u \neq v \Rightarrow f(u) \neq f(v)$ for any u, v in the domain of f . Equivalently: each output comes from at most one input.

Is $f(x) = x^2$, $f: \mathbb{R} \rightarrow \mathbb{R}$, one-to-one? No

The notes answer: No, $f(-1) = f(1)$. Indeed $-1 \neq 1$ but both give the output 1.

Theorem (p. 46, Lay §1.9)

Let $T(x) = Ax$ where A is an $m \times n$ matrix. Then the following are equivalent:

- (1) T is one-to-one;
- (2) $Ax = b$ has at most one solution (for every b in $\mathbb{R}^m$);
- (3) $Ax = 0$ only has the zero solution;
- (4) each column of A is pivotal.

Proof (p. 46, with the “already know” part cited)

(2) $\iff$ (3) $\iff$ (4): *already known*. (3) $\iff$ (4) is the Theorem on p. 22 (9/1 notes: $Ax = 0$ has a nonzero solution $\iff A$ has a column without a pivot). (2) $\iff$ (4): by the Theorem on p. 10 (8/27 notes), when $Ax = b$ has a solution, it is unique $\iff$ every column of the RREF is pivotal, i.e. there are no free variables. So with every column pivotal there is never more than one solution; and if some column is not pivotal, then already $b = 0$ (which always has the solution $x = 0$) has infinitely many solutions.

(1) $\iff$ (2). Since $T(x) = Ax$, statement (2) says that $T(x) = b$ has at most one solution x for each b , i.e. each output comes from at most one input. That is what one-to-one means.

Examples (our addition).

- The map of p. 39 with $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$: the RREF $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ has both columns pivotal, so T is one-to-one (this is why $x = (2, 5)$ on p. 40 was the *only* solution). The third row has no pivot, so T is not onto. ✓
- Rotation: both columns pivotal (previous section), so it is one-to-one as well as onto.
- Projection onto the x_1 -axis: column 2 is not pivotal, so it is not one-to-one; indeed $T(0, 1) = T(0, 0) = (0, 0)$. ✓

Looking ahead. The next lecture (9/8) starts with an example of exactly this kind: a map $\mathbb{R}^2 \rightarrow \mathbb{R}^3$, its standard matrix, and the questions “one-to-one? onto?”. Rows decide onto; columns decide one-to-one.

Summary

Fact	Formula / test
Matrix map	$T(x) = Ax, \mathbb{R}^n \rightarrow \mathbb{R}^m$ for A of size $m \times n$
Image of $x \mapsto Ax$	$\text{span}\{a_1, \dots, a_n\}$ (columns of A)
Linear	$T(x + y) = T(x) + T(y), T(cx) = cT(x)$
Linear $\Rightarrow$	$T(\sum c_i v_i) = \sum c_i T(v_i)$, and $T(0) = 0$
Standard basis	$e_i = \text{column } i \text{ of } I_n; Ae_i = \text{column } i \text{ of } A$
Standard matrix	$A = [T(e_1) \cdots T(e_n)]$, then $T(x) = Ax$
Rotation by φ	$\begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix}$
b in the image	$Ax = b$ consistent
Onto	pivot in every row of A
One-to-one	pivot in every column of $A \iff Ax = 0$ only for $x = 0$

Corrections to the handwritten notes

Page	As written	Should be	Why
44	“for every b in $\mathbb{R}^n$ there is x in $\mathbb{R}^n$ ”	for every b in $\mathbb{R}^m$	b lives in the codomain $\mathbb{R}^m$
46	(2) “ $Ax = b$ has at most one solution”	... for every b	the statement needs the quantifier; for one fixed b it does not imply (1)