

Study Guide for Test 2, Math 1564, Fall 2026

The background for this study guide is homeworks 6, 7, 8, 9 and the notes 9/22–10/13 (recursions, linear maps and their matrices, Markov chains, determinants, and eigenvalues as roots of $\det(A - \lambda I)$). Each problem mixes ideas from several homeworks; the line under each problem says which ones. Justify your answers. If you claim something is true, prove it; if false, give a counterexample.

1. Let $T : \mathbb{R}_{\leq 2}[x] \rightarrow \mathbb{R}_{\leq 2}[x]$ be the shift $T(f)(x) = f(x + 1)$, and let $\beta = \{1, x, x^2\}$.
 - a Find the matrix A that represents T in the basis β (same basis in the domain and codomain).
 - b Compute $\det A$. Is T invertible? Is it volume-preserving? Find A^{-1} in two ways: by computing it from A , and as the matrix of the map $f(x) \mapsto f(x - 1)$. Are the entries of A^{-1} integers, and is that what you expected?
 - c Find all eigenvalues of A and all eigenvectors (solve $(A - \lambda I)v = 0$). Does $\mathbb{R}^3$ have a basis of eigenvectors of A ? Interpret the eigenvectors as polynomials.
 - d Show that T maps $\{f : f(0) = 0\}$ onto $\{g : g(-1) = 0\}$, and use this to write down a basis of $\{g \in \mathbb{R}_{\leq 2}[x] : g(-1) = 0\}$.

Builds on: HW6 #3 (shift maps, kernels of evaluation), HW7 #2 (matrix of a map in bases), HW9 #3 (integer matrices), 10/8 (eigenvalues of triangular matrices).

2. Consider the stochastic matrix

$$P = \begin{bmatrix} 1/2 & 1/4 & 0 \\ 1/2 & 1/2 & 1/2 \\ 0 & 1/4 & 1/2 \end{bmatrix}.$$

- a Check that P is stochastic and find its steady state q .
- b Is P regular? Is q a sink? Compute $\lim_{k \rightarrow \infty} P^k$.
- c Find all eigenvalues of P by solving $\det(P - \lambda I) = 0$ (*hint:* expand along row 1 and factor out $\frac{1}{2} - \lambda$), and an eigenvector for each. Check that the eigenvectors form a basis of $\mathbb{R}^3$. Write $e_1 = q + c_2 v_2 + c_3 v_3$ in this eigenbasis and give an exact formula for $P^k e_1$, $k \geq 1$. Check it for $k = 1, 2$.
- d Without computing anything, explain why $\det(P - I) = 0$ for *every* stochastic matrix P . (*Hint:* add up the rows of $P - I$.)
- e Let $Q = \frac{1}{2}(P + I)$. Show Q is stochastic and has the same steady states as P .

Builds on: HW8 #1, #2, #5 (sinks, limits of P^k , mixtures), 9/29 (a basis of eigenvectors solves $X_{k+1} = AX_k$), 10/8 (row operations and $\det$, $\det(A - \lambda I) = 0$).

3. Consider the recursion $x_{k+2} = x_{k+1} + c x_k$, where $c \neq 0$ is a real number.

- a For $c = 6$, find a basis of the solution space, and the solution with $x_0 = 1$, $x_1 = 8$. Show that $x_k/3^k$ converges and find the limit.

- b For $c = 6$, write the recursion as $X_{k+1} = AX_k$. Compute $\det(\lambda I - A)$ and $\det A$. Is X_k determined by X_{k+1} ?
- c For which c does the characteristic equation have a repeated root? Give a basis of the solution space in that case.
- d For which c are the roots non-real? Give a basis of *real* solutions for $c = -1$.
- e Why is $c = 0$ excluded? What is the dimension of the solution space when $c = 0$?

Builds on: HW6 #1 (Fibonacci, growth rate), HW7 #1, #4 (dimension, matrix form, present determines past), HW9 #4 (characteristic polynomial of a companion matrix), 9/24 (three cases).

4. Let $V = M_{2 \times 2}(\mathbb{R})$ and $T : V \rightarrow V$, $T(A) = A + A^\top$.

- a Show that T is linear.
- b Find $\ker T$ and the image of T , and their dimensions. Check the rank theorem.
- c Find the matrix M of T in the basis $E_{11}, E_{12}, E_{21}, E_{22}$. What is $\det M$, and why could you predict it from (b)?
- d Show that every symmetric matrix is an eigenvector of T with eigenvalue 2, and every nonzero skew-symmetric matrix is an eigenvector with eigenvalue 0. Is there a basis of V consisting of eigenvectors of T ? How does this relate to writing $A = A_+ + A_-$?
- e Show that the determinant of a 2×2 skew-symmetric matrix is ≥ 0 , and that every 3×3 skew-symmetric matrix is singular.

Builds on: HW6 #2 (symmetric/skew-symmetric subspaces, $A = A_+ + A_-$), HW9 #1 (det of transposes), 9/29 (eigenvectors).

5. Let $T : \mathbb{R}_{\leq 2}[x] \rightarrow \mathbb{R}^3$, $T(f) = (f(0), f(1), f(2))^\top$.

- a Find the matrix V of T in the basis $\{1, x, x^2\}$ and the standard basis.
- b Compute $\det V$ by row reduction. Is T invertible? Is it volume-preserving?
- c Find the polynomial f of degree ≤ 2 with $f(0) = 1$, $f(1) = 0$, $f(2) = 3$. Check the coefficient of x^2 with Cramer's rule.
- d Without computing V^{-1} , explain why V^{-1} cannot have all integer entries. Then compute V^{-1} and confirm.
- e Let $U : \mathbb{R}_{\leq 2}[x] \rightarrow \mathbb{R}^2$, $U(f) = (f(0), f(1))^\top$. Find $\dim \ker U$ and a basis of $\ker U$. Is U onto?

Builds on: HW7 #3 (evaluation at points), HW6 #3 (kernels of evaluation maps), HW9 #3 (integer inverses), 10/13 (Cramer's rule).

6. Let

$$M = \begin{bmatrix} 2 & 1 & 5 & 7 \\ 1 & 1 & -3 & 4 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}.$$

- a Compute $\det M$ using the block formula, and check it by row reduction.
- b Explain why M^{-1} has integer entries. Verify that $M^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{bmatrix}$ and compute the two diagonal blocks.
- c Let $S(x) = Ax$ on $\mathbb{R}^2$. What is the area of the image of the unit disk? Of the unit square?
- d For which integers k is $x \mapsto \begin{bmatrix} k & 1 \\ 1 & 1 \end{bmatrix} x$ area-preserving?

Builds on: HW9 #2 (block matrices), HW9 #3 (volume-preserving integer matrices), 10/13 (area and $|\det|$).

7. Consider $x_{k+3} - 2x_{k+2} - x_{k+1} + 2x_k = 0$.

- a Factor $r^3 - 2r^2 - r + 2$ and find a basis of the solution space. Prove the basis vectors are independent.
- b Find the solution with $x_0 = 0$, $x_1 = 0$, $x_2 = 6$. Check it against the recursion for $k = 3$.
- c Write the recursion as $X_{k+1} = AX_k$ and compute $\det(\lambda I - A)$ by cofactor expansion. What is $\det A$?
- d Use the recursion backwards to find x_{-1} for the solution in (b), and check it with your formula.
- e Find $\lim_{k \rightarrow \infty} x_k / 2^k$.

Builds on: HW7 #4 (order-3 recursion, distinct roots, invertible companion matrix), HW7 #3c (Vandermonde), HW9 #4 (characteristic polynomial), HW6 #1b (growth rate).

8. True or false? Prove or give a counterexample.

- a If $A^\top = A^{-1}$ and $\det A = -1$, then $A + I$ is not invertible.
- b The $n \times n$ matrices with $\det A = 0$ form a subspace of $M_{n \times n}(\mathbb{R})$.
- c $W = \{f \in \mathbb{R}_{\leq 3}[x] : f(1) = f(-1)\}$ is a subspace of dimension 3. (If true, find a basis.)
- d If A has integer entries and $\det A = 2$, then A^{-1} cannot have all integer entries.
- e If A is invertible and λ is an eigenvalue of A , then $\lambda \neq 0$ and $1/\lambda$ is an eigenvalue of A^{-1} .
- f A positive stochastic matrix has exactly one steady state.
- g If (x_k) solves $x_{k+2} + a_1x_{k+1} + a_0x_k = 0$ with $a_0 \neq 0$, and $x_5 = x_6 = 0$, then $x_k = 0$ for all $k \geq 0$.
- h There is a real 3×3 matrix A with $A^2 = -I$.
- i If P and Q are stochastic, then $\frac{1}{3}P + \frac{2}{3}Q^2$ is stochastic.

Builds on: all four homeworks; see the table in the answer key.