

Test 2 Study Guide — Answer Key

Math 1564, Fall 2026

Covers homeworks 6–9 and the notes 9/22–10/13

Where each homework shows up

Homework problem	Idea	Study guide
HW6 #1	Fibonacci, $c_{\pm}$, growth rate	3a, 7e
HW6 #2	symmetric/skew subspaces, $A = A_+ + A_-$	4
HW6 #3	evaluation maps, kernels, shift isomorphism	1d, 5e, 8c
HW7 #1	dimension of solutions of $X_{k+1} = AX_k$	3b
HW7 #2	matrix of a linear map in bases	1a, 4c, 5a
HW7 #3	evaluation at points, Vandermonde	5, 7a
HW7 #4	order-3 recursion, invertible companion matrix	7, 3b, 8e, 8g
HW8 #1	$\lim P^k = [q \cdots q]$	2b
HW8 #2	steady state, regular $\Rightarrow$ sink	2a–c
HW8 #3, #4	constant-column matrices, convex sets of steady states	2b remark, 2e, 8f
HW8 #5	products and mixtures of stochastic matrices	2e, 8i
HW9 #1	$\det A^T = \det A$ tricks (orthogonal, skew, $A^2 = -I$)	4e, 8a, 8h
HW9 #2	block determinants	6a, 6b
HW9 #3	integer matrices, $ \det = 1$, integer inverses	1b, 5d, 6b, 6d, 8d
HW9 #4	characteristic polynomial of companion matrix	3b, 7c

Facts used most often.

- **(F1)** The matrix of T in bases β, γ has i -th column $[T(v_i)]_{\gamma}$ (9/29).
- **(F2)** Triangular: $\det =$ product of diagonal entries; eigenvalues = diagonal entries (10/8).
- **(F3)** $\det(AB) = \det A \det B$, $\det A^T = \det A$, $\det(cA) = c^n \det A$ for $n \times n$ (10/8).
- **(F4)** $(A^{-1})_{ij} = C_{ji} / \det A$ (10/13). Hence an integer matrix has an integer inverse $\iff \det A = \pm 1$ (HW9 #3).
- **(F5)** If $v_1, \dots, v_n$ is a basis of eigenvectors ($Av_i = \lambda_i v_i$) and $X_0 = c_1 v_1 + \cdots + c_n v_n$, then $A^k X_0 = c_1 \lambda_1^k v_1 + \cdots + c_n \lambda_n^k v_n$ (9/29).
- **(F6)** Regular $\Rightarrow$ sink $\Rightarrow$ unique steady state, and $P^k \rightarrow [q \cdots q]$ (10/1, HW8).

Problem 1 (shift map)

1a

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

By (F1), apply T to each basis vector and read off coordinates:

$$T(1) = 1, \quad T(x) = x + 1, \quad T(x^2) = (x + 1)^2 = 1 + 2x + x^2,$$

so the columns are $(1, 0, 0)$, $(1, 1, 0)$, $(1, 2, 1)$. Check on $f = 3 + 5x + x^2$: $A(3, 5, 1)^T = (9, 7, 1)^T$, and $f(x + 1) = 3 + 5x + 5 + x^2 + 2x + 1 = 9 + 7x + x^2$. ✓

1b

$$\det A = 1: \text{invertible and volume-preserving}; A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

A is upper triangular with 1's on the diagonal, so $\det A = 1$ (F2). Hence T is invertible, and volume-preserving since $|\det A| = 1$.

From A : back-substitution (or the cofactor formula) gives the matrix above. Check: row 1 of A times the columns of A^{-1} gives 1, $-1 + 1 = 0$, $1 - 2 + 1 = 0$; rows 2 and 3 similarly give $(0, 1, 0)$ and $(0, 0, 1)$. ✓

As the inverse shift: $S(f)(x) = f(x - 1)$ undoes T (HW6 #3d). $S(1) = 1$, $S(x) = x - 1$, $S(x^2) = 1 - 2x + x^2$, giving the columns $(1, 0, 0)$, $(-1, 1, 0)$, $(1, -2, 1)$: the same matrix.

Integer entries, as expected: A has integer entries and $\det A = 1$, so by HW9 #3a (F4) A^{-1} must have integer entries.

1c

Only eigenvalue 1; eigenvectors = nonzero multiples of $(1, 0, 0)$; no eigenbasis

$$A \text{ is triangular with diagonal } 1, 1, 1, \text{ so } 1 \text{ is the only eigenvalue (F2). } A - I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}:$$

row 2 gives $x_3 = 0$, then row 1 gives $x_2 = 0$; x_1 is free. So the eigenvectors are exactly the nonzero multiples of e_1 . Any two of them are dependent, so there is no basis of $\mathbb{R}^3$ made of eigenvectors, and the 9/29 method (F5) can't be used to compute A^k here.

As polynomials: e_1 is the constant polynomial 1. So $f(x + 1) = \lambda f(x)$ (with $f \neq 0$, $\deg f \leq 2$) only for constant f and $\lambda = 1$, which makes sense: a nonconstant polynomial can't repeat its

values after a shift by 1.

1d

T maps $\text{Ker}(\text{ev}_0)$ onto $\text{Ker}(\text{ev}_{-1})$; basis $\{x+1, (x+1)^2\}$

Into: if $f(0) = 0$ then $g = T(f)$ has $g(-1) = f(-1+1) = f(0) = 0$.

Onto: if $g(-1) = 0$, put $f(x) = g(x-1)$. Then $f(0) = g(-1) = 0$ and $T(f)(x) = f(x+1) = g(x)$.

T is linear and one-to-one (it's invertible by 1b), so it is an isomorphism between these subspaces. It sends the basis $\{x, x^2\}$ of $\{f : f(0) = 0\}$ (HW6 #3c) to the basis $\{T(x), T(x^2)\} = \{x+1, (x+1)^2\}$ of $\{g : g(-1) = 0\}$. This is exactly HW6 #3e with $a = -1$.

Problem 2 (Markov chain)

2a

$$q = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$$

Entries are ≥ 0 ; column sums are $\frac{1}{2} + \frac{1}{2} + 0 = 1$, $\frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$, $0 + \frac{1}{2} + \frac{1}{2} = 1$.

$Pq = q$ with $q = (x, y, z)$: row 1 gives $\frac{x}{2} + \frac{y}{4} = x$, so $y = 2x$; row 3 gives $\frac{y}{4} + \frac{z}{2} = z$, so $y = 2z$.

Row 2 then holds automatically. So $q = (x, 2x, x)$, and $x + 2x + x = 1$ gives $x = \frac{1}{4}$. Check:

$$Pq = (\frac{1}{8} + \frac{1}{8}, \frac{1}{2}(\frac{1}{4} + \frac{1}{2} + \frac{1}{4}), \frac{1}{8} + \frac{1}{8}) = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4}). \checkmark$$

2b

Regular ($P^2 > 0$), so q is a sink and $P^k \rightarrow \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$

$$P^2 = \begin{bmatrix} 3/8 & 1/4 & 1/8 \\ 1/2 & 1/2 & 1/2 \\ 1/8 & 1/4 & 3/8 \end{bmatrix}$$

has all entries positive, so P is regular, so it has a sink, which must be the steady state q (F6).

By HW8 #1, column i of P^k is $P^k e_i \rightarrow q$, so $P^k \rightarrow [q \ q \ q]$.

Remark (HW8 #3). The limit $L = [q \ q \ q]$ has every column equal to q , so $LX = q$ for every probability vector X : the limit matrix sends everything straight to the steady state in one step.

2c

Eigenvalues $\frac{1}{2}$ and 0; $P^k e_1 = q + \left(\frac{1}{2}\right)^{k+1} (1, 0, -1)$ for $k \geq 1$

Eigenvalues (10/8). Expand $\det(P - \lambda I)$ along row 1:

$$\left(\frac{1}{2} - \lambda\right) \left[\left(\frac{1}{2} - \lambda\right)^2 - \frac{1}{8}\right] - \frac{1}{4} \cdot \frac{1}{2} \left(\frac{1}{2} - \lambda\right) = \left(\frac{1}{2} - \lambda\right) \left[\left(\frac{1}{2} - \lambda\right)^2 - \frac{1}{4}\right] = \left(\frac{1}{2} - \lambda\right) (-\lambda)(1 - \lambda).$$

So the eigenvalues are $1, \frac{1}{2}, 0$. (The last step uses $a^2 - \frac{1}{4} = (a - \frac{1}{2})(a + \frac{1}{2})$ with $a = \frac{1}{2} - \lambda$.)

- $\lambda = \frac{1}{2}$: $P - \frac{1}{2}I = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{4} & 0 \end{bmatrix}$ gives $y = 0, x = -z$: $v_2 = (1, 0, -1)$. Check: $Pv_2 = (\frac{1}{2}, 0, -\frac{1}{2}) = \frac{1}{2}v_2$. ✓
- $\lambda = 0$: Nul P : $\frac{x}{2} + \frac{y}{4} = 0$ and $\frac{y}{4} + \frac{z}{2} = 0$ give $y = -2x, z = x$: $v_3 = (1, -2, 1)$. Check: $Pv_3 = 0$. ✓

Basis check. Using $4q = (1, 2, 1)$:

$$\det \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} = 1(0 - 2) - 1(2 + 2) + 1(-2 - 0) = -8 \neq 0,$$

so $\{q, v_2, v_3\}$ is a basis of $\mathbb{R}^3$.

Coordinates of e_1 . $e_1 - q = (\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}) = c_2(1, 0, -1) + c_3(1, -2, 1)$. The middle entry gives $c_3 = \frac{1}{4}$, the first gives $c_2 = \frac{1}{2}$, and the third checks: $-\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$. ✓ So $e_1 = q + \frac{1}{2}v_2 + \frac{1}{4}v_3$.

Powers (F5). $P^k e_1 = q + \frac{1}{2} \left(\frac{1}{2}\right)^k v_2 + \frac{1}{4} \cdot 0^k v_3 = q + \left(\frac{1}{2}\right)^{k+1} (1, 0, -1)$ for $k \geq 1$.

Check: $k = 1$: $(\frac{1}{4} + \frac{1}{4}, \frac{1}{2}, \frac{1}{4} - \frac{1}{4}) = (\frac{1}{2}, \frac{1}{2}, 0)$, column 1 of P . ✓ $k = 2$: $(\frac{3}{8}, \frac{1}{2}, \frac{1}{8})$, column 1 of P^2 . ✓

Why the coefficient of q is exactly 1: v_2 and v_3 have entries summing to 0, while e_1 and q have entries summing to 1. This is why every probability vector converges to q itself, not a multiple of it.

2d

The rows of $P - I$ add up to the zero row

Entry j of (row 1 + $\dots$ + row n) of $P - I$ is (column j sum of P) $- 1 = 1 - 1 = 0$. Adding rows $2, \dots, n$ to row 1 is a sequence of row replacements, which don't change $\det$ (10/8), and produces a zero row. So $\det(P - I) = 0$.

Consequence: 1 is an eigenvalue of every stochastic matrix. (That an eigenvector can be chosen with nonnegative entries, i.e. a steady state, is a harder fact, which the homework let you assume.)

2e

Q is stochastic with the same steady states

$Q = \frac{1}{2}P + \frac{1}{2}I$ is HW8 #5c with $t = \frac{1}{2}$ and the stochastic matrix I , so it is stochastic. And $Qq = q \iff \frac{1}{2}Pq + \frac{1}{2}q = q \iff Pq = q$.

Problem 3 (second-order recursion with a parameter)

The characteristic equation of $x_{k+2} - x_{k+1} - cx_k = 0$ is $r^2 - r - c = 0$, with discriminant $1 + 4c$.

3a

$x_k = 2 \cdot 3^k - (-2)^k$, and $x_k/3^k \rightarrow 2$

$c = 6$: $r^2 - r - 6 = (r - 3)(r + 2)$. Two distinct real roots, so the basis is $\{(3^k), ((-2)^k)\}$ (9/24). From $c_1 + c_2 = 1$ and $3c_1 - 2c_2 = 8$: $5c_1 = 10$, so $c_1 = 2$ and $c_2 = -1$. Check: $x_2 = x_1 + 6x_0 = 14$ and $2 \cdot 9 - 4 = 14$. ✓
 $x_k/3^k = 2 - (-\frac{2}{3})^k \rightarrow 2$, since $|\frac{2}{3}| < 1$. So x_k grows like $2 \cdot 3^k$ (as in HW6 #1b).

3b

$A = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}$, $\det(\lambda I - A) = \lambda^2 - \lambda - 6$, $\det A = -6$; yes

With $X_k = (x_k, x_{k+1})$: the top entry of X_{k+1} is x_{k+1} (row $(0, 1)$), and the bottom is $x_{k+2} = 6x_k + x_{k+1}$ (row $(6, 1)$).

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 \\ -6 & \lambda - 1 \end{bmatrix} = \lambda(\lambda - 1) - 6 = \lambda^2 - \lambda - 6,$$

the characteristic polynomial of the recursion, as HW9 #4 predicts. $\det A = -6 \neq 0$, so A is invertible and $X_k = A^{-1}X_{k+1}$: the present determines the past (HW7 #4b). The solution space has dimension 2 (HW7 #1).

3c

$c = -\frac{1}{4}$; basis $\{(\frac{1}{2})^k\}, \{k(\frac{1}{2})^k\}$

A repeated root needs $1 + 4c = 0$, i.e. $c = -\frac{1}{4}$, with root $r = \frac{1}{2}$. By the repeated-root case (9/24) the basis is $\{(\frac{1}{2})^k, (k\frac{1}{2})^k\}$. Check the second: $(\frac{1}{2})^k [\frac{k+2}{4} - \frac{k+1}{2} + \frac{k}{4}] = (\frac{1}{2})^k \cdot \frac{k+2-2k-2+k}{4} = 0$. ✓

3d

Non-real roots $\iff c < -\frac{1}{4}$; for $c = -1$: $(1, \frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, \dots)$ and $(0, 1, 1, 0, -1, -1, \dots)$

The roots are non-real exactly when $1 + 4c < 0$. For $c = -1$ the equation is $r^2 - r + 1 = 0$ with roots $r_{\pm} = \frac{1 \pm i\sqrt{3}}{2}$. A basis of real solutions is $(\operatorname{Re} r_+^k), (\operatorname{Im} r_+^k)$ (9/24).

Using $r^2 = r - 1$: $r^3 = r^2 - r = -1$, so the powers repeat with period 6. Taking real and imaginary parts:

$$(\operatorname{Re} r_+^k) = (1, \frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, \dots), \quad (\operatorname{Im} r_+^k) = \frac{\sqrt{3}}{2}(0, 1, 1, 0, -1, -1, \dots).$$

Scaling the second by $\frac{2}{\sqrt{3}}$ gives the nicer basis vector $(0, 1, 1, 0, -1, -1, \dots)$. Check with $x_{k+2} = x_{k+1} - x_k$: $\frac{1}{2} - 1 = -\frac{1}{2}$, $-\frac{1}{2} - \frac{1}{2} = -1$; and $1 - 0 = 1$, $1 - 1 = 0$, $0 - 1 = -1$. ✓

3e

$c = 0$ gives a “shorter” recursion

With $c = 0$ the recursion is $x_{k+2} = x_{k+1}$: the term x_k drops out ($a_0 = 0$), so it is really order 1 and the solutions are constant from x_1 on. The companion matrix $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ has determinant 0, so the past is *not* determined: x_0 is never constrained. Counting sequences $(x_k)_{k \geq 0}$, the solution space has dimension 2 (x_0 and x_1 free), but one of those dimensions is the meaningless free x_0 . This is the degenerate case the assumption $a_0 \neq 0$ rules out (9/29).

Problem 4 ($T(A) = A + A^\top$)

4a

Linear

$T(cA + dB) = (cA + dB) + (cA + dB)^\top = c(A + A^\top) + d(B + B^\top) = cT(A) + dT(B)$, using $(cA + dB)^\top = cA^\top + dB^\top$.

4b

Ker T = skew-symmetric (dim 1); image = symmetric (dim 3)

$T(A) = 0 \iff A^\top = -A$, so Ker T is the skew-symmetric matrices, with basis $\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$: dimension $\frac{2 \cdot 1}{2} = 1$ (HW6 #2a).

Image $\subseteq$ **symmetric**: $(A + A^\top)^\top = A^\top + A$. **Symmetric** $\subseteq$ **image**: if $S^\top = S$ then $T(\frac{1}{2}S) = \frac{1}{2}S + \frac{1}{2}S = S$. So the image is the symmetric matrices: dimension $\frac{2 \cdot 3}{2} = 3$.
 Rank theorem: $1 + 3 = 4 = \dim M_{2 \times 2}$. ✓

4c

$$M = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \det M = 0$$

$T(E_{11}) = 2E_{11}$, $T(E_{12}) = E_{12} + E_{21} = T(E_{21})$, $T(E_{22}) = 2E_{22}$. Coordinates in $(E_{11}, E_{12}, E_{21}, E_{22})$ give the columns above.

$\det M = 0$ (columns 2 and 3 are equal). Predictable from (b): $\text{Ker } T \neq \{0\}$, so T is not one-to-one, so M is not invertible.

4d

Yes: $\{E_{11}, E_{22}, E_{12} + E_{21}\}$ (eigenvalue 2) $\cup \{E_{12} - E_{21}\}$ (eigenvalue 0)

If $S^\top = S$ then $T(S) = 2S$; if $K^\top = -K$ then $T(K) = 0 = 0 \cdot K$. So every nonzero symmetric matrix is an eigenvector with eigenvalue 2, and every nonzero skew one is an eigenvector with eigenvalue 0. (Equivalently, their coordinate vectors are eigenvectors of M .)

The four matrices listed are bases of the symmetric (dim 3) and skew (dim 1) subspaces. Together they are independent: if $s + k = 0$ with s symmetric and k skew, then $s = -k$ is both, so $s = k = 0$ (HW6 #2b), and each group's coefficients vanish. Four independent vectors in the 4-dimensional V form a basis.

Relation to HW6 #2c: writing $A = A_+ + A_-$ is writing A in this eigenbasis. Then

$$T(A) = 2A_+ + 0 \cdot A_- = 2A_+,$$

and indeed $A + A^\top = 2 \cdot \frac{A + A^\top}{2}$. ✓

4e

$$\det \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = b^2 \geq 0; \text{ every } 2 \times 2 \text{ skew matrix is singular}$$

A 2×2 skew matrix has zero diagonal ($a_{ii} = -a_{ii}$) and $a_{21} = -a_{12}$, so it is $\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$, with $\det = 0 - (-b^2) = b^2 \geq 0$.

3×3 : $\det A = \det A^\top = \det(-A) = (-1)^3 \det A = -\det A$ (F3), so $\det A = 0$. (This is HW9 #1b with $n = 3$.)

Problem 5 (evaluation at 0, 1, 2)

5a

$$V = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

Column i is $T(x^{i-1})$: $T(1) = (1, 1, 1)$, $T(x) = (0, 1, 2)$, $T(x^2) = (0, 1, 4)$. Row j lists the powers of the j -th point, as in HW7 #3a.

5b

$\det V = 2$: invertible, not volume-preserving

$R_2 - R_1$, $R_3 - R_1$, then $R_3 - 2R_2$ (row replacements only):

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad \det V = 1 \cdot 1 \cdot 2 = 2.$$

$\det V \neq 0$, so T is invertible (as HW7 #3c predicts: the points 0, 1, 2 are distinct). $|\det V| = 2 \neq 1$, so not volume-preserving.

5c

$$f = 1 - 3x + 2x^2$$

With $f = a + bx + cx^2$: $f(0) = a = 1$; $f(1) = 1 + b + c = 0$; $f(2) = 1 + 2b + 4c = 3$. So $b + c = -1$ and $b + 2c = 1$, giving $c = 2$, $b = -3$. Check: $f(2) = 1 - 6 + 8 = 3$. ✓

Cramer's rule for c (third coordinate), with $b = (1, 0, 3)$:

$$c = \frac{\det V_3(b)}{\det V} = \frac{1}{2} \det \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix} = \frac{1 \cdot (3 - 0) - 0 + 1 \cdot (2 - 1)}{2} = \frac{4}{2} = 2. \quad \checkmark$$

5d

$\det V^{-1} = \frac{1}{2}$ is not an integer

If V^{-1} had integer entries, its determinant would be an integer (cofactor expansion only adds and multiplies entries). But $\det V^{-1} = 1/\det V = \frac{1}{2}$. (This is HW9 #3a.) Computing:

$$V^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 2 & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}.$$

Check: row 3 of V times column 1: $1 - 3 + 2 = 0$; times column 3: $0 - 1 + 2 = 1$. ✓

Meaning: column j of V^{-1} lists the coefficients of the quadratic that is 1 at the j -th point and 0 at the others. For example, column 1 is $1 - \frac{3}{2}x + \frac{1}{2}x^2 = \frac{(x-1)(x-2)}{2}$.

5e

$\dim \text{Ker } U = 1$, basis $\{x^2 - x\}$; U is onto

U has matrix $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, with 2 pivots and 3 columns: rank 2, so $\dim \text{Ker } U = 3 - 2 = 1$.

The kernel is the polynomials vanishing at 0 and 1: $f(0) = a = 0$ and $f(1) = b + c = 0$, so $f = c(x^2 - x) = c x(x - 1)$.

Onto: there is a pivot in every row (or: $a + (b - a)x$ has values a at 0 and b at 1). This is the same structure as HW6 #3b-c.

Problem 6 (block matrix, areas)

6a

$\det M = \det A \det C = 1 \cdot 1 = 1$

$\det A = 2 \cdot 1 - 1 \cdot 1 = 1$ and $\det C = 3 \cdot 1 - 1 \cdot 2 = 1$, so $\det M = 1$ by HW9 #2.

Row reduction: $R_2 - \frac{1}{2}R_1$ and $R_4 - \frac{2}{3}R_3$ (replacements) make M upper triangular with diagonal $2, \frac{1}{2}, 3, \frac{1}{3}$. The product is $2 \cdot \frac{1}{2} \cdot 3 \cdot \frac{1}{3} = 1$. ✓

6b

Integer inverse; $A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, C^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$

M has integer entries and $\det M = 1$, so by (F4) every entry $C_{ji}/1$ of M^{-1} is an integer (HW9 #3a).

Block formula: multiplying blockwise,

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \begin{bmatrix} A^{-1} & X \\ 0 & C^{-1} \end{bmatrix} = \begin{bmatrix} I & AX + BC^{-1} \\ 0 & I \end{bmatrix},$$

which is I exactly when $X = -A^{-1}BC^{-1}$. The 2×2 inverse formula gives the two diagonal blocks above. (For completeness, $-A^{-1}BC^{-1} = \begin{bmatrix} -2 & -1 \\ 13 & -14 \end{bmatrix}$, all integers.)

6c

Disk: area π . **Square:** area 1

Area scales by $|\det A| = 1$ (10/13). So the image of the unit disk (an ellipse) has area $1 \cdot \pi = \pi$, and the image of the unit square (the parallelogram on $(2, 1)$ and $(1, 1)$) has area $|2 - 1| = 1$.

6d

$k = 0$ or $k = 2$

$\det \begin{bmatrix} k & 1 \\ 1 & 1 \end{bmatrix} = k - 1$, and area-preserving means $|k - 1| = 1$, so $k = 2$ ($\det 1$) or $k = 0$ ($\det -1$, orientation-reversing).

Problem 7 (third-order recursion)

7a

$(r - 1)(r + 1)(r - 2)$; basis $\{(1), ((-1)^k), (2^k)\}$

Factor by grouping: $r^3 - 2r^2 - r + 2 = r^2(r - 2) - (r - 2) = (r - 2)(r^2 - 1) = (r - 1)(r + 1)(r - 2)$.

As in HW7 #4a, each (r^k) with $r \in \{1, -1, 2\}$ is a solution. **Independence:** a combination vanishing for $k = 0, 1, 2$ gives $M^\top c = 0$, where M is the Vandermonde matrix of HW7 #3c with distinct points $1, -1, 2$. It is invertible, so $c = 0$. Three independent solutions in the

3-dimensional solution space form a basis.

7b

$$x_k = -3 + (-1)^k + 2^{k+1}$$

$$x_k = c_1 + c_2(-1)^k + c_3 2^k \text{ with}$$

$$c_1 + c_2 + c_3 = 0, \quad c_1 - c_2 + 2c_3 = 0, \quad c_1 + c_2 + 4c_3 = 6.$$

The third minus the first gives $3c_3 = 6$, so $c_3 = 2$. Then $c_1 + c_2 = -2$ and $c_1 - c_2 = -4$, so $c_1 = -3$ and $c_2 = 1$.

Check $k = 0, 1, 2$: $-3 + 1 + 2 = 0$, $-3 - 1 + 4 = 0$, $-3 + 1 + 8 = 6$. ✓ And $k = 3$: the recursion gives $x_3 = 2x_2 + x_1 - 2x_0 = 12$; the formula gives $-3 - 1 + 16 = 12$. ✓

7c

$$\det(\lambda I - A) = \lambda^3 - 2\lambda^2 - \lambda + 2; \det A = -2$$

$x_{k+3} = -2x_k + x_{k+1} + 2x_{k+2}$, so $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix}$. Expanding along column 1 (only two nonzero entries):

$$\det \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 2 & -1 & \lambda - 2 \end{bmatrix} = \lambda \det \begin{bmatrix} \lambda & -1 \\ -1 & \lambda - 2 \end{bmatrix} + 2 \det \begin{bmatrix} -1 & 0 \\ \lambda & -1 \end{bmatrix} = \lambda(\lambda^2 - 2\lambda - 1) + 2,$$

which is $\lambda^3 - 2\lambda^2 - \lambda + 2$, as HW9 #4 says. Setting $\lambda = 0$: $\det(-A) = (-1)^3 \det A = 2$, so $\det A = -2$.

7d

$$x_{-1} = -3$$

At $k = -1$ the recursion reads $x_2 - 2x_1 - x_0 + 2x_{-1} = 0$, so $x_{-1} = \frac{-x_2 + 2x_1 + x_0}{2} = \frac{-6}{2} = -3$. (This works because $a_0 = 2 \neq 0$, i.e. $\det A \neq 0$.) Check with the formula: $-3 + (-1)^{-1} + 2^0 = -3 - 1 + 1 = -3$. ✓

7e

$$\lim x_k/2^k = 2$$

$x_k/2^k = -\frac{3}{2^k} + \left(-\frac{1}{2}\right)^k + 2 \rightarrow 0 + 0 + 2 = 2$. The largest root, 2, controls the growth, just as r_+ did for Fibonacci (HW6 #1b).

Problem 8 (true/false)

(a) True

$A + I = A + AA^\top = A(I + A^\top) = A(I + A)^\top$. Taking determinants (F3): $\det(A + I) = \det A \cdot \det(A + I) = -\det(A + I)$, so $\det(A + I) = 0$. (Compare HW9 #1a: $A^\top = A^{-1}$ forces $\det A = \pm 1$.) Example: $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $A + I = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

(b) False

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ have $\det = 0$, but their sum I has $\det = 1$. Not closed under addition.

(c) True; basis $\{1, x^2, x^3 - x\}$

W is the kernel of the linear map $f \mapsto f(1) - f(-1)$ (HW6 #3 style). For $f = a + bx + cx^2 + dx^3$, $f(1) - f(-1) = 2b + 2d$. This map is onto $\mathbb{R}$, so $\dim W = 4 - 1 = 3$. W is the set where $b = -d$: $f = a + cx^2 + d(x^3 - x)$, so $\{1, x^2, x^3 - x\}$ spans W , and it is independent (different leading terms).

(d) True

A^{-1} with integer entries would have an integer determinant, but $\det A^{-1} = \frac{1}{2}$.

(e) True

If $\lambda = 0$, then $Av = 0$ with $v \neq 0$, so $\text{Nul } A \neq \{0\}$ and A is not invertible (9/29). So $\lambda \neq 0$. Applying A^{-1} to $Av = \lambda v$ gives $v = \lambda A^{-1}v$, so $A^{-1}v = \frac{1}{\lambda}v$. (For a companion matrix with $a_0 \neq 0$ this is why the roots are nonzero and the recursion runs backwards, HW7 #4b.)

(f) True

Positive means P^1 has all entries positive, so P is regular, so it has a sink, which is its only steady state (F6, 10/1). Compare HW8 #4: otherwise there would be infinitely many.

(g) True

$X_5 = (x_5, x_6) = 0$. The companion matrix is invertible ($a_0 \neq 0$), so $X_k = A^{k-5}X_5 = 0$ for every k , including $k < 5$ (negative powers). So every $x_k = 0$. ("The present determines the past", HW7 #4b.)

(h) False

$(\det A)^2 = \det(A^2) = \det(-I) = (-1)^3 = -1$, impossible for a real number $\det A$. (HW9 #1c with $n = 3$.)

(i) True

Q^2 is stochastic by HW8 #5a, and $\frac{1}{3}P + \frac{2}{3}Q^2$ is stochastic by HW8 #5c with $t = \frac{1}{3}$.